MATHEMATICAL MODELING FOR MISSILE SYSTEMS

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ABSTRACT

Documentary quotations from well-known men in their fields are presented to yield some insight into the nature of mathematical models, simulation, operations research, dynamic programming, cybernetics, etc. Following these general concepts which are a prerequisite to an understanding of the current weapon system jargon, an airborne weapon system is broken down into the three major subsystems:

1. Airframe and environment.

2. Control major subsystem.

3. Guidance major subsystem.

One of the major efforts in the scientific synthesis and analysis of a physical system is the obtaining of a realistic mathematical model of the physically realized system. This is especially true of an advanced weapon system which is being designed against a futuristic target (not yet in existence). The mathematical model (which must be computable) provides a flexible economic means of predicting system behavior for various environmental and tactical conditions.

The mathematical models are considered as two types:

1. Deterministic models.

2. Probabilistic models.

Vector methods are used to develop the air frame equations of motion, gimballed body kinematics, rotating spheroidal earth equations and rotating spheroidal earth geometry.

Transformation matrices for various Euler angle sequences are developed, means of generating the direction cosines and a number of systems of equations of airframe motion are presented.

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PART I

INTRODUCTION

THE GUIDED MISSILE WEAPON SYSTEM IS ENTERING ITS SECOND DECADE OF POPULARITY IN OUR LABORATORIES THROUGHOUT THE COUNTRY. THE EARLIER DESIGN AND DEVELOPMENT PROCEEDED ALONG RATHER UNCERTAIN LINES RELYING VERY HEAVILY ON TRIAL AND ERROR, WITH VERY LITTLE DETAILED OVER-ALL SYSTEM THEORETICAL STUDIES. THE PRESENT STATE-OF-THE-ART STILL RELIES MUCH TOO HEAVILY ON THE TRIAL AND ERROR APPROACH. HOWEVER, IT IS HOPED THAT WITHIN THE NEXT DECADE THE CURRENT INVESTIGATIONS INTO WEAPON SYSTEMS EVALUATION, "OPTIMIZATION THROUGH MULTI-DECISION PROCESSES," ETC., WILL HAVE YIELDED FRUITFUL RESULTS.

THE UNDERSTANDING OF A COMPLEX PHYSICAL PROCESS UNDER VARIOUS ENVIRONMENTAL AND OPERATING CONDITIONS REQUIRES TERRIFIC TECHNICAL EFFORT . SINCE MANY WEAPON SYSTEMS ARE DESIGNED AGAINST FUTURE TARGETS NOT YET BUILT, ONE MUST RESORT TO AN EXTRAPOLATION OR APPROXIMATION PROCESS THROUGH MODELS. THE MATHEMATICAL MODEL IS THE MOST SOPHISTICATED AND DESIRABLE TYPE OF MODEL TO PREDICT SYSTEM BEHAVIOR.

THE OVERWHELMING MAJORITY OF THE PAST ANALYTICAL METHODS UTILIZED IN SYSTEM DESIGN AND ANALYSIS HAVE BEEN BASED UPON THE SO CALLED RIGID MATHEMATICAL MODEL THAT IS THE DETERMINISTIC MODEL. THIS INCLUDES CLASSICAL MECHANICS, CIRCUIT THEORY ETC. EVEN THOUGH THE VARIABILITY OF THE THOUSANDS OF MISSILE SYSTEM PARAMETERS AND VARIABLES HAVE BEEN RECOGNIZED IN THE PAST, VERY LITTLE HAS BEEN DONE ABOUT IT BECAUSE OF THE MATHEMATICAL LIMITATIONS. EVEN IF THE MATHEMATICS DEFINING SO COMPLEX A STOCHASTIC PROBLEM WERE AVAILABLE MOST PRESENT DAY COMPUTERS ARE INADEQUATE TO HANDLE SUCH A TASK.

CONSEQUENTLY TWO MAJOR PARALLEL EFFORTS ARE TAKING PLACE. ONE IS THE DEVELOPMENT OF LARGER AND BETTER COMPUTING FACILITIES WHICH ARE THE PHYSICAL BUILDING BLOCK FROM WHICH THE MATHEMATICAL MODEL IS TRANSFORMED BACK INTO A PHYSICAL SYSTEM MODEL. SECOND IS THE DEVELOPMENT OF PROBABILISTIC MATHEMATICAL MODELS WHICH REALISTICALLY REPRESENT THE WEAPON SYSTEM BEHAVIOR.

THE MATHEMATICAL DERIVATIONS IN THIS REPORT ARE FOR THE DEVELOPMENT OF THE DETERMINISTIC MODEL. THE PRIMARY OBJECTIVE OF A WEAPON SYSTEM IS TO VECTOR AN INTERCEPTOR SUFFICIENTLY CLOSE TO A TARGET SUCH THAT THE LETHALITY OF THE WARHEAD WITH RESPECT TO THE SUSCEPTIBILITY OF THE TARGET WILL CONSTITUTE A "KILL." THE FIRST REQUIREMENT IS A GEOMETRICAL PROBLEM, CONSEQUENTLY THE MATHEMATICAL STATEMENTS OF THE GUIDANCE AND CONTROL PHILOSOPHIES ARE THE THEORETICAL DESIGN EQUATIONS, WHICH ARE TO BE INSTRUMENTED AND MECHANIZED INTO A PHYSICALLY REALIZED SYSTEM.

ACCEPTING THE GEOMETRICAL ASPECTS AS A BASIC REQUIREMENT FOR AN UNDERSTANDING OF AIR BORNE WEAPON SYSTEMS, THE DETERMINISTIC MATHEMATICAL MODELS OF AIRFRAME DYNAMICS, GIMBALLED BODY KINEMATICS, ROTATING SPHEROIDAL EARTH GEOMETRY ARE DERIVED BY VECTORS.

THE MAJORITY OF THE REFERENCE FRAMES USED ARE ORTHONORMAL BASIS VECTORS.

BECAUSE OF THE NICE PROPERTIES OF THE ORTHONORMAL BASIS VECTORS THE DEVELOPMENT OF THE ORTHONORMAL TRANSFORMATIONS ARE HANDLED SLIGHTLY DIFFERENTLY THAN NORMALLY ENCOUNTERED IN THE INTRODUCTORY TEST BOOKS. IT IS FELT THAT THIS APPROACH MORE READILY YIELDS A DEEPER PHYSICAL INSIGHT INTO THE THREE DIMENSIONAL GEOMETRICAL PROBLEMS.

EACH MISSILE SYSTEM REQUIRES FROM FIVE TO TEN OR MORE SETS OF BASIS VECTORS. CONSEQUENTLY BEFORE DISCUSSING ANY SYSTEM PHYSICAL QUANTITIES SUCH AS VELOCITIES, POSITIONS, FORCES, ETC., THE ORIENTATION OF THE VARIOUS SYSTEM BASIS VECTORS ARE DEFINED. SINCE THE METRIC OF THE SPACE SPANNED BY THE ORTHONORMAL BASIS VECTORS IS SIMPLE $R_i \cdot R_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$

THE NEXT RELATIONSHIPS REQUIRED IS THE RELATIONSHIP BETWEEN THE VARIOUS BASIS VECTORS, FOR EXAMPLE $\overline{R}_i \cdot \overline{B}_j = R^B_{ij} = B^R_{ji}$

WHICH YIELDS THE DIRECTION COSINE MATRIX BETWEEN THE DIFFERENT SETS OF BASIS VECTORS.

AFTER THE ABOVE RELATIONSHIPS ON THE BASIS VECTORS ARE ESTABLISHED, THE PHYSICAL QUANTITIES OF THE SYSTEM, WHICH ARE VECTORS, CAN BE HANDLED WITHOUT ANY FURTHER ADO AS TO THE ORIENTATIONS OF THE MANY COMPONENTS.

THE DETERMINISTIC MATHEMATICAL MODELS DEFINING THE GUIDANCE AND CONTROL PHILOSOPHIES WHICH ARE TO BE INSTRUMENTED, AND THE AIRFRAME EQUATIONS OF MOTION WHICH ARE TO BE SIMULATED ARE NOT UNIQUE. THUS, IT IS SHOWN BY EXAMPLES HOW THE AIRFRAME EQUATIONS OF MOTION CAN BE WRITTEN IN A NUMBER OF WAYS. ONE OF THE CRITERION IN THE SELECTION OF A MATHEMATICAL MODEL IS THAT THE MODEL VARIABLES CORRESPOND TO THE MEASURABLE VARIABLES OF THE PHYSICAL SYSTEM.

WEAPON SYSTEM CONCEPTS

OF THE MULTITUDE OF COMPLEX CONTROL SYSTEMS IN OUR SOCIETY, SUCH AS THE TELEPHONE SYSTEMS, AUTOMATED INDUSTRIAL PROCESSES, WEAPON SYSTEMS, ETC., THE WEAPON SYSTEM IN PARTICULAR MAY BE SINGLED OUT AS A SYSTEM WHICH IT IS DESIRED WILL ALWAYS BE IN THE SIMULATION STAGE. THAT IS, THAT IT WILL NEVER HAVE TO BE UTILIZED IN ITS DESIGN ENVIRONMENT, WAR.

THUS, FROM THIS BROADER SIMULATION CONCEPT, THE BREAD-BOARD SUB-SYSTEM EXPERIMENT IN THE LABORATORY, THE RANGE FLIGHT TESTS, TO THE PLANNED MILITARY MANEUVERS UTILIZING AIR DEFENSE MISSILES AGAINST TARGET DRONES ARE EXAMPLES OF PHYSICAL SIMULATION. ALL OF THE INTERVENING SYNTHESIS AND ANALYSIS IN THE DESIGN AND EVALUATION OF A WEAPON SYSTEM ARE EITHER PURELY PHYSICAL SIMULATION, PURELY MATHEMATICAL OR VARIOUS DEGREES OF COMBINED PHYSICAL AND MATHEMATICAL SIMULATION. THE VARIABLES IN THE PHYSICAL AND MATHEMATICAL SIMULATION MAY BE CONTINUOUS OR DISCRETE OR COMBINATIONS THEREOF.

Simulation may be considered as:

1. Physical simulation.

2. Mathematical simulation.

3. Combined physical and mathematical simulation.

Mathematical simulation is based on the mechanizing or programming on computers of defining equations. These equations are ideal theoretical equations or equations based on empirical data. The ultimate system simulation equations should reflect the characteristics (transfer functions, noise etc.) of the actual instrumentation developed.

Before discussing weapon systems as such, a few general system concepts which are in current usage will be considered merely to indicate the challenging scope and magnitude of the problems encountered.

Operations Research

Operations research is a very broad field which is not too easily defined. It is an applied science utilizing all of the academic disciplines, primarily the social, biological, and physical sciences and its purpose is to provide some quantitative measure as a basis for decisions. Goode and Machol[1] state that the fundamental difference between operations research and systems engineering is that the former is interested in making procedural changes while the latter is interested in making equipment changes.

Dynamic Programming

Is a mathematical theory for handling problems involving multi-stage decision processes. Bellman treats two major types of problems in his book Dynamic Programming[2]. First he considers a multi-stage allocation process of deterministic type which from the mathematical point of view is a multidimensional maximization problem which is related to the calculus of variations. Second he considers a multi-stage decision process of stochastic type.

Bellman states that dynamic programming is designed to treat multistage processes possessing certain invariant aspects, whereas the theory of linear programming is designed to treat processes possessing certain features of linearity.

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[1] By permission from Control Systems Engineering by H. H. Goode and R. E. Machol. Copyright, 1957, McGraw-Hill Book Company, Inc.

[2] By permission from Dynamic Programming by R. Bellman. Copyrighted 1957. Princeton University Press.

Linear Programming

Vajda[3] states that linear programming deals with a maximizing (or minimizing) problem that cannot be solved by the methods of calculus and that the theory of games can be shown to form a special case of those of linear programming.

Relaxation Methods

"The word 'relaxation' has at least two meanings to engineers and mathematicians. In its narrow sense, it denotes a class of iterative methods for solving a system of linear algebraic equations. This class of methods is characterized by the successive reduction to zero of the worst residual.

In its broad sense, relaxation includes the approximate reformulation of broad classes of physical problems as systems of linear equations to be solved. In this sense, relaxation has come to connote a large part of the art of computing. The art of computing is the adaptation of all sorts of numerical techniques, proved or unproved, to the end result of getting answers to numerical problems.

Besides the art of computing, there is a science of numerical analysis, whose practitioners survey and prove theorems about known computing techniques, devise new ones, and test methods in cooperation with computing artists. Contemporary numerical analysis is often oriented toward the exploitation of the new electronic digital computers."[4]

THE NATURE OF MONTE CARLO[5]

"Monte Carlo methods are distinguished by their experimental nature. Whether or not a particular experimental method is eligible to be called a Monte Carlo method may be largely a matter of personal taste. Monte Carlo methods are not known for all problems nor do specific problems necessarily admit a unique specific Monte Carlo approach. On the contrary, there may exist different Monte Carlo methods for a given problem, not obviously related one to another.

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[3] By permission from The Theory of Games and Linear Programming by S. Vajda. Copyrighted 1957. Methuen and Company, Ltd.

[4] By permission from Modern Mathematics For The Engineer by E. F. Beckenbach. Copyright 1956. McGraw-Hill Book Company, Inc.

[5] Ibid.

"For definiteness, we shall class as a Monte Carlo method any procedure which involves the use of statistical sampling techniques to approximate the solution of a mathematical or physical problem. For a homely illustration, consider the veteran poker player who knows (at least subconsciously) the approximate probability of filling an 'inside straight,' having learned it the hard way. The required probability could, of course, be calculated directly but it is likely that most poker players have learned by experience rather than by direct calculation. In such instances the player may be said to have profited by a Monte Carlo approach.

"Monte Carlo methods have an ancient and honorable history, with recorded examples dating back at least a few centuries. The early pioneers in the theory of probability were aware of the relationship of certain functional equations to the solutions of probability problems arising from the analysis of games of chance. More recently, there have been known to the physicists similar relationships between probabilistic models of particle behavior and various classical functional equations. Most commonly, the solution of these problems has proceeded, by classical methods of analysis, from the functional equation to which the probabilistic model reduces. The term 'Monte Carlo' was popularized, in very recent years, mostly in connection with problems of statistical mechanics, particle diffusion, and the like, when it was observed that in cases which did not yield readily to classical analysis the numerical computation of approximate solutions could sometimes be approached through the probabilistic model. The analytical model was by-passed by the process of tracing the histories of numbers of individual particles, each exhibiting random behavior in accordance with the probabilistic model, with the random elements provided by a chance device or by reference to tables of random numbers. Each particle history might be considered a single experiment, the accuracy of the result depending on the statistical approximation obtained from the aggregation of many such experiments.

"The classical direction of analysis was thus reversed, with the computation proceeding, in first applications, in direct analogy to standard statistical models suggested by the physical problems. Since then much ingenuity has been applied to the modification of the statistical models and to the invention of new models, not strictly analogous to the physical problem, in order to permit more economical calculation of approximate solutions to the original problem. Stimulated by these applications, mathematicians have begun to devote considerable attention to the search for probability models which will permit the approximate solution of functional equations of various types, without reference to underlying physical systems. Methods exist for certain classes of partial differential equations, integral equations, eigenvalue problems, and simultaneous linear equations."

Quoting from Meyer at a Symposium on Monte Carlo Methods:[6] "The word probability has not always meant exactly the same thing to those who use and to those who write about probability. It is not strange, then, the term 'Monte Carlo Methods' should likewise be subject to various interpretations."

Householder indicates that the novelty of the Monte Carlo method lies in the suggestion that where an equation arising in a nonprobabilistic context demands a numerical solution not easily obtainable by standard numerical methods, there may exist a stochastic process with distributions or parameters which satisfy the equation, and it may actually be more efficient to construct such a process and compute the statistics than attempt to use those standard methods. These suggestions seem to have been due to Ulam, von Neumann and Enrico Fermi..

Householder points out that the problem encountered when one is given an equation is whether or not there is a stochastic process which yields a distribution such that it, or some set of its parameters, satisfies the equation. If so, what is the efficient method of obtaining the statistics and assessing them? He further states that the method is probably never efficient for yielding an entire distribution unless the distribution is obtained only by integrating out other variables, and that basically the method is one of numerical integration.

According to Weiner, the theory of the transmission of messages in electrical engineering is a part of a larger field which includes not only the study of language but the study of messages as a means of controlling machinery and society, the development of computing machines and other such automats, certain reflections upon psychology and the nervous system, and a tentative new theory of scientific method. This larger theory of messages according to Weiner is a probabilistic theory, an intrinsic part of the movement that owes its origin to Willard Gibbs.

Weiner states that the act of giving an order to a machine is not essentially different from the act of giving an order to a person, since as far as his consciousness is concerned he is aware of the order that has gone out and the signal of compliance that has come back. To him personally, the fact that the signal in its intermediate stages has gone through a machine rather than through a person is irrelevant and does not greatly change his relation to the signal. Thus, he claims, the theory of control in engineering, whether human or animal or mechanical, is a chapter in the theory of messages.

Weiner points out that there are detailed differences in messages and in problems of control, not only between a living organism and a machine, but within each narrower class of beings. Weiner coined the word Cybernetics

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[6] By permission from Symposium on Monte Carlo Methods by H. A. Meyer. Copyrighted 1956. John Wiley and Sons, Inc.

AND POINTS OUT THAT ITS PURPOSE IS TO DEVELOP A LANGUAGE AND TECHNIQUES THAT WILL ENABLE US TO ATTACK THE PROBLEM OF CONTROL AND COMMUNICATION IN GENERAL, AND ALSO TO FIND THE PROPER REPERTORY OF IDEAS AND TECHNIQUES TO CLASSIFY THEIR PARTICULAR MANIFESTATIONS UNDER CERTAIN CONCEPTS.

MATHEMATICAL MODEL

DR. BELLMAN OF THE RAND CORPORATION VERY NICELY STATES THE ROLE OF THE MATHEMATICAL MODEL.[7]

"THE GOAL OF THE SCIENTIST IS TO COMPREHEND THE PHENOMENA OF THE UNIVERSE HE OBSERVES AROUND HIM. TO PROVE THAT HE UNDERSTANDS, HE MUST BE ABLE TO PREDICT, AND TO PREDICT, ONE REQUIRES QUANTITATIVE MEASUREMENTS. A QUALITATIVE PREDICTION SUCH AS THE OCCURRENCE OF AN ECLIPSE OR AN EARTHQUAKE OR A DEPRESSION SOMETIME IN THE NEAR FUTURE DOES NOT HAVE THE SAME SATISFYING FEATURES AS A SIMILAR PREDICTION ASSOCIATED WITH A DATE AND TIME, AND PERHAPS BACKED UP BY THE OFFER OF SIDE WAGER.

"TO PREDICT QUANTITATIVELY ONE MUST HAVE A MECHANISM FOR PRODUCING NUMBERS, AND THIS NECESSARILY ENTAILS A MATHEMATICAL MODEL. IT SEEMS REASONABLE TO SUPPOSE THAT THE MORE REALISTIC THIS MATHEMATICAL MODEL, THE MORE ACCURATE THE PREDICTION.

"THERE IS, HOWEVER, A POINT OF DIMINISHING RETURNS. THE ACTUAL WORLD IS EXTREMELY COMPLICATED, AND AS A MATTER OF FACT THE MORE THAT ONE STUDIES IT THE MORE ONE IS FILLED WITH WONDER THAT WE HAVE EVEN "ORDER OF MAGNITUDE" EXPLANATIONS OF THE COMPLICATED PHENOMENA THAT OCCUR, MUCH LESS FAIRLY CONSISTENT "LAWS OF NATURE." IF WE ATTEMPT TO INCLUDE TOO MANY FEATURES OF RELAITY IN OUR MATHEMATICAL MODEL, WE FIND OURSELVES ENGULFED BY COMPLICATED EQUATIONS CONTAINING UNKNOWN PARAMETERS AND UNKNOWN FUNCTIONS. THE DETERMINATION OF THESE FUNCTIONS LEADS TO EVEN MORE COMPLICATED EQUATIONS WITH EVEN MORE UNKNOWN PARAMETERS AND FUNCTIONS, AND SO ON. TRULY A TALE THAT KNOWS NO END.

"IF, ON THE OTHER HAND, MADE TIMID BY THESE PROSPECTS, WE CONSTRUCT OUR MODEL IN TOO SIMPLE A FASHION, WE SOON FIND THAT IT DOES NOT PREDICT TO SUIT OUR TASTES.

"IT FOLLOWS THAT THE SCIENTIST, LIKE THE PILGRIM, MUST WEND A STRAIGHT AND NARROW PATH BETWEEN THE PITFALLS OF OVERSIMPLIFICATION AND THE MORASS OF OVERCOMPLICATION.

"KNOWING THAT NO MATHEMATICAL MODEL CAN YIELD A COMPLETE DESCRIPTION OF REALITY, WE MUST RESIGN OURSELVES TO THE TASK OF USING A SUCCESSION OF MODELS OF GREATER AND GREATER COMPLEXITY IN OUR EFFORTS TO UNDERSTAND. IF WE

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[7] BY PERMISSION FROM DYNAMIC PROGRAMMING BY R. BELLMAN. COPYRIGHTED 1957, PRINCETON UNIVERSITY PRESS.

OBSERVE SIMILAR STRUCTURAL FEATURES POSSESSED BY THE SOLUTIONS OF A SEQUENCE OF MODELS, THEN WE MAY FEEL THAT WE HAVE AN APPROXIMATION TO WHAT IS CALLED A LAW OF NATURE . . . .

IT FOLLOWS THAT FROM A TELEOLOGICAL POINT OF VIEW THE PARTICULAR NUMERICAL SOLUTION OF ANY PARTICULAR SET OF EQUATIONS IS OF FAR LESS IMPORTANCE THAN THE UNDERSTANDING OF THE NATURE OF THE SOLUTION, WHICH IS TO SAY THE INFLUENCE OF THE PHYSICAL PROPERTIES OF THE SYSTEM UPON THE FORM OF THE SOLUTION."

BECKENBACH STATES:[8] "MANY SCIENTIFIC INVESTIGATIONS ARE BASED ON THE ISOMORPHISM, OR SAMENESS OF STRUCTURE, OF DIFFERENT PHYSICAL AND MATHEMATICAL SYSTEMS. THIS IS PARTICULARLY TRUE OF THE SUBJECTS TREATED IN THE PRESENT BOOK, WITH THEIR SOMETIMES MORE AND SOMETIMES LESS PRECISE MATHEMATICAL MODELS. AT TIMES THE MODELS HAVE BEEN CONSIDERABLY ALTERED THROUGH TRANSFORMATIONS BUT THE NEW MODELS HAVE REMAINED VALUABLE BECAUSE OF THE INVARIANCE OF CERTAIN OF THEIR ASPECTS UNDER THE TRANSFORMATIONS."

ACCORDING TO JOHN HAMMERSLEY OF TRINITY COLLEGE, OXFORD; VERY FEW MATH TEACHERS HAVE EVER USED MATHEMATICS IN PRACTICE, AND THE MATHEMATICAL EXAMINATION PROBLEMS ARE USUALLY CONSIDERED UNFAIR IF INSOLUBLE OR IMPROPERLY DESCRIBED. WHEREAS THE MATHEMATICAL PROBLEMS OF REAL LIFE ARE ALMOST INVARIABLY INSOLUBLE AND BADLY STATED, AT LEAST IN THE FIRST INSTANCE. HE STATES THAT IN REAL LIFE, THE MATHEMATICIAN'S MAIN TASK IS TO FORMULATE PROBLEMS BY BUILDING AN ABSTRACT MATHEMATICAL MODEL CONSISTING OF EQUATIONS, WHICH SHALL BE SIMPLE ENOUGH TO SOLVE WITHOUT BEING SO CRUDE THAT THEY FAIL TO MIRROR REALITY. HAMMERSLEY CONSIDERS THE TASK OF SOLVING EQUATIONS AS A MINOR TECHNICAL MATTER COMPARED WITH THIS FASCINATING AND SOPHISTICATED CRAFT OF MODEL-BUILDING, WHICH CALLS FOR BOTH CLEAR, KEEN COMMON SENSE AND THE HIGHEST QUALITIES OF ARTISTIC AND CREATIVE IMAGINATION.

THE INCREASING COMPLEXITY OF WEAPON SYSTEMS NECESSITATES THE TEAM APPROACH AS EXPRESSED BY REESE:[9] "UNLESS A MAN ENJOYS WORKING WITH OTHERS, UNLESS HE IS INTERESTED IN CONSIDERING OTHER PEOPLE'S PROBLEMS, UNLESS HE FINDS IT INTERESTING TO EVOLVE THE APPROPRIATE MATHEMATICAL MODEL FOR HANDLING SITUATIONS THAT ARE OFTEN NOT CORRECTLY OR CLEARLY DESCRIBED, AND TO BRING HIS MATHEMATICAL MATURITY TO BEAR ON SITUATIONS HE HAS NOT HIMSELF SELECTED HE PROBABLY DOES NOT BELONG IN INDUSTRY. ONE RELEVANT RESULT OF THE SURVEY WAS NOT A SURPRISE, BUT THE STATISTICS DO SERVE TO CORROBORATE A STRONG PRIOR IMPRESSION. GROUP RESEARCH IS VIEWED MORE FAVORABLY BY INDUSTRIAL MATHEMATICIANS, AND THERE IS MORE OF IT AMONG THEM THAN AMONG UNIVERSITY PEOPLE . . . .

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[8] BY PERMISSION FROM MODERN MATHEMATICS FOR THE ENGINEER BY E. F. BECKENBACH. COPYRIGHTED 1956. MCGRAW-HILL BOOK COMPANY, INC.

[9] BY PERMISSION FROM MATHEMATICIANS IN THE MARKETPLACE BY N. REES. COPYRIGHTED 1958. AMERICAN MATHEMATICAL MONTHLY, VOL. 65, NR 5, MAY 1958.

"The application of Boolean algebra to computer design, of group theory to communications theory, of number theory to numerical analysis, and of noncommutative algebra and group theory to nuclear physics leaves one with the quite correct impression that almost any mathematics is applicable, and that broadly educated mathematicians are needed . . . .

"I have chosen to quote from a British report to emphasize the similarity of the situation in England and the United States. In our country, too, a sampling of problems confronted in industry shows a broad spectrum of mathematical disciplines in use in the attempt to find solutions. On a recent visit to the Bell Telephone Laboratories, I was told of a few of the problems in which some members of the mathematical research group are interested. One of these originated in the Financial Department of the American Telephone and Telegraph Company, and concerned the charges for private wire services. The mathematical result that was derived at Bell Laboratories gave a new theorem in the theory of trees. Interestingly enough the same result seems to have been found at about the same time at the Mathematical Centrum in Amsterdam when, in the design of a new computer, the old computer was asked to determine what wiring would keep the total length of wire as small as possible. In the Telephone Company's problem, the Federal Communications Commission requires that the charges be based on the shortest network that can be constructed connecting plants at several locations; in the computer problem, the effort is to minimize the network of wires connecting a collection of pins. The theorem previously existing in the literature required that the minimum distance between any two points be found --- a requirement poorly adapted to computation. The new result gives a systematic procedure that can easily be translated into a machine program. We have here a situation that illustrates the new requirement often found in the industrial applications of mathematics - to find a new theorem which will enable us to exploit the speed of electronic computers. For industrial applications, the computability of a method is even more important than elegance is for proof in pure mathematics.

"Another Bell Laboratories problem concerns communications in the presence of noise, and involves quantizing the sending of bits of information. The problem is to translate into usable form an existence proof and asymptotic theorems due to Shannon. Though the problem is unsolved, some progress has been made using the theory of groups . . . .

"A new kind of demand for mathematicians has been produced by the current expansion of the use of operations research in industry. This is a type of non-academic employment that relies less on specific techniques than do the traditional fields of applied mathematics. A large expansion in this type of activity is in process, with a significant increase in the opportunities for mathematicians . . . .

"The training needed for effective work as an industrial methematician has been described to me by Hendrik Bode, on the basis of his experience at Bell Telephone Laboratories. In his judgment, an applied mathematician needs to serve one, or perhaps several, internships. For example, in these

DAYS OF AUTOMATICALLY SEQUENCED COMPUTING MACHINES A THOROUGH ACQUAINTANCE WITH NUMERICAL METHODS, STANDARD APPROXIMATING PROCEDURES, AND THE VARIOUS WAYS OF ATTACKING PROBLEMS NUMERICALLY ARE OBVIOUSLY OF SPECIAL IMPORTANCE. THIS IS THE SORT OF ABILITY WHICH IS BEST LEARNED BY ACTUAL PRACTICE IN A COMPUTING CENTER. THE COMPUTING FACILITY NEED NOT BE A LARGE ONE - IN FACT THERE MAY BE A POSITIVE ADVANTAGE IN HAVING TO USE ONE'S INGENUITY TO OVERCOME MECHANICAL INADEQUACIES; BUT A FIRST-HAND ACQUAINTANCE WITH NUMBERS, OF THE SORT THAT IS DEVELOPED ONLY BY A CERTAIN AMOUNT OF LABORATORY WORK, IS ALMOST ESSENTIAL.

"ANOTHER SIGNIFICANT INTERNSHIP COULD BE SPENT IN A STATISTICAL CENTER. THIS WOULD GIVE THE ASPIRING APPLIED MATHEMATICIAN A BETTER APPRECIATION THAN HE COULD PROBABLY HAVE OTHERWISE OF THE RELIABILITY OF THE FACTUAL STATEMENTS TO WHICH HE WILL BE SUBJECTED LATER, AND WOULD ALSO GIVE HIM A FEELING FOR THE PROBABLE SIGNIFICANCE, OR LACK OF SIGNIFICANCE, OF A GIVEN ELEMENT OF A LOGICAL FRAME-WORK IN A PARTICULAR SITUATION.

"ANOTHER PROFITABLE, IF BRIEF, INTERNSHIP MIGHT BE SERVED AS A PARTICIPANT IN AN ACTUAL FUNCTIONING APPLIED MATHEMATICS GROUP IN AN INDUSTRIAL OR SIMILAR ENVIRONMENT. THIS WOULD BE PARTICULARLY USEFUL IF IT OCCURRED BEFORE THE END OF THE FORMAL TRAINING PERIOD. IT MIGHT, OF COURSE, BE POSSIBLE TO FIND WAYS OF MEETING MOST, IF NOT ALL, OF THESE INTERNSHIP REQUIREMENTS IN A SINGLE PLACE."

GOODE AND MACHOL[10] DISTINGUISH BETWEEN FOUR DIFFERENT TYPES OF MATHEMATICAL MODELS:

1. THE ANALYTICAL RIGID MODEL.

2. THE NUMERICAL RIGID MODEL.

3. THE ANALYTICAL PROBABILITY MODEL.

4. THE NUMERICAL PROBABILITY MODEL OR MONTE CARLO MODEL.

THE FIRST TWO TYPES ARE WELL KNOWN AND OF THE NATURE OF OHM'S AND NEWTON'S LAWS, HOWEVER, THE LAST TWO TYPES ARE NOT AS FAMILIAR, AS WELLER POINTS OUT:[11] "A SOMEWHAT DIFFERENT SOURCE OF MATHEMATICAL COMPULSION IS DERIVED FROM A CONSIDERATION OF THE STATISTICAL NATURE OF CERTAIN PHYSICAL SITUATIONS. STUDENTS OF THE LIFE SCIENCES HAVE LONG USED STATISTICAL TECHNIQUES BECAUSE THE SYSTEMS UNDER STUDY WERE EXCEEDINGLY COMPLEX AND BECAUSE

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MANY VARIABLES AFFECTING THEM COULD NOT BE CONTROLLED. ENGINEERS, ON THE OTHER HAND, HAVE, IN THE PAST, BEEN LARGELY CONCERNED WITH DETERMINISTIC PROBLEMS. BUT THE COMING OF COMPLEXITY IN TECHNOLOGICAL DEVELOPMENT HAS LED MORE AND MORE TO CONSIDERATIONS OF STOCHASTIC RATHER THAN DETERMINISTIC BEHAVIOR."

GOODE AND MACHOL CLASS PROBABILITY AND SIMULATION AS THE BASIC TOOLS OF SYSTEM DESIGN.[12] "THERE IS AN EXTENSIVE LITERATURE ON THE SUBJECT OF THE LOGICAL BASES OF PROBABILITY. THERE ARE SEVERAL APPROACHES TO THIS PHILOSOPHICAL QUESTION. ONE, REPRESENTED BY REICHENBACK AND VON MISES, ATTEMPTS TO DEFINE PROBABILITY ON A FREQUENCY BASES; THAT IS, IF THE NUMBER OF EXPERIMENTS IS ALLOWED TO APPROACH INFINITY, THEN THE PROBABILITY OF A FAVORABLE OUTCOME MAY BE DEFINED AS THE LIMIT OF THE PROPORTION OF THE EXPERIMENTS WHICH ARE FAVORABLE. A SECOND APPROACH, REPRESENTED BY CARNAP, JEFFREYS, AND KEYNES, VIEWS PROBABILITY AS A LOGICAL RELATION ANALOGOUS TO THAT OF LOGICAL IMPLICATION BUT ADMITTING OF DEGREES. A THIRD APPROACH, REPRESENTED BY KOOPMAN AND KOLMOGOROFF, ATTEMPTS TO DEFINE PROBABILITY ON AN AXIOMATIC BASIS; IT STATES THAT PROBABILITY IS A GAME TO BE PLAYED ACCORDING TO CERTAIN RULES, WORKED OUT ON A STRICT MATHEMATICAL BASIS. OUR OWN USE WILL BE MORE LIKE THE LAST."

GOODE AND MACHOL STATE THAT PROBABILITY IS THE DEDUCTIVE SCIENCE OF CHANCE IN THAT IT PREDICTS THE OUTCOME RESULTING FROM A SET OF ASSUMPTIONS. WHEREAS STATISTICS IS THE INDUCTIVE SCIENCE OF CHANCE IN THAT IT ENABLES ONE TO MAKE INFERENCES ABOUT THE NATURE OF THE UNDERLYING DISTRIBUTION AND ESTIMATES OF ITS PARAMETERS FROM KNOWLEDGE OF THE OUTCOMES OF EXPERIMENTS.

WIENER HAS THE FOLLOWING TO SAY ABOUT PROBABILITY AND STATISTICS:
. . . AS STATISTICAL THEORY IS ESSENTIALLY A BRANCH OF PROBABILITY THEORY AND AS PROBABILITY THEORY HAS BEEN REDUCED TO THE THEORY OF LEBESQUE MEASURE AND LEBESQUE INTEGRATION BY A SERIES OF WRITERS INCLUDING KOLMOGOROFF, KNINTCHINE, CRAMER, DOOB, LEVY, AND THE AUTHOR, ...

M. LOEVE[13] ON PROBABILISTIC MODELS STATES: "PROBABILITY THEORY IS CONCERNED WITH THE MATHEMATICAL ANALYSIS OF THE INTUITIVE NOTION OF 'CHANCE' OR 'RANDOMNESS,' WHICH, LIKE ALL NOTIONS, IS BORN OF EXPERIENCE. THE QUANTITATIVE IDEA OF RANDOMNESS FIRST TOOK FORM AT THE GAMING TABLES, AND PROBABILITY THEORY BEGAN, WITH PASCAL AND FERMAT (1654), AS A THEORY OF GAMES OF CHANCE. SINCE THEN, THE NOTION OF CHANCE HAS FOUND ITS WAY INTO ALMOST ALL BRANCHES OF KNOWLEDGE. IN PARTICULAR, THE DISCOVERY THAT PHYSICAL 'OBSERVABLES,' EVEN THOSE WHICH DESCRIBE THE BEHAVIOR OF ELEMENTARY PARTICLES, WERE

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TO BE CONSIDERED AS SUBJECT TO LAWS OF CHANCE MADE AN INVESTIGATION OF THE NOTION OF CHANCE BASIC TO THE WHOLE PROBLEM OF RATIONAL INTERPRETATION OF NATURE.

"A THEORY BECOMES MATHEMATICAL WHEN IT SETS UP A MATHEMATICAL MODEL OF THE PHENOMENA WITH WHICH IT IS CONCERNED, THAT IS, WHEN, TO DESCRIBE THE PHENOMENA, IT USES A COLLECTION OF WELL-DEFINED SYMBOLS AND OPERATIONS ON THE SYMBOLS. AS THE NUMBER OF PHENOMENA, TOGETHER WITH THEIR KNOWN PROPERTIES, INCREASES, THE MATHEMATICAL MODEL EVOLVES FROM THE EARLY CRUDE NOTIONS UPON WHICH OUR INTUITION WAS BUILT IN THE DIRECTION OF HIGHER GENERALITY AND ABSTRACTNESS.

IN THIS MANNER, THE INNER CONSISTENCY OF THE MODEL OF RANDOM PHENOMENA BECAME DOUBTFUL, AND THIS FORCED A REBUILDING OF THE WHOLE STRUCTURE IN THE SECOND QUARTER OF THIS CENTURY, STARTING WITH A FORMULATION IN TERMS OF AXIOMS AND DEFINITIONS. THUS, THERE APPEARED A BRANCH OF PURE MATHEMATICS PROBABILITY THEORY --CONCERNED WITH THE CONSTRUCTION AND INVESTIGATION PER SE OF THE MATHEMATICAL MODEL OF RANDOMNESS."

PROFESSOR BARTLETT[14] OF THE UNIVERSITY OF MANCHESTER STATES IN HIS BOOK INTRODUCTION TO STOCHASTIC PROCESSES: "IN THIS BOOK WE ARE GOING TO CONSIDER A SUBJECT WHICH IN PARTICULAR APPLICATIONS HAS ARISEN SINCE THE BEGINNINGS OF PROBABILITY THEORY, BUT THE SYSTEMATIC TREATMENT OF WHICH HAS ONLY RECENTLY BEGUN TO RECEIVE THE ATTENTION IT DESERVES. WE MAY, ROUGHLY SPEAKING, THINK OF THIS SUBJECT AS THE 'DYNAMIC' PART OF STATISTICAL THEORY, OR THE STATISTICS OF 'CHANGE,' IN CONTRAST WITH THE 'STATIC' STATISTICAL PROBLEMS WHICH HAVE HITHERTO BEEN THE MORE SYSTEMATICALLY STUDIED. BY A STOCHASTIC PROCESS WE SHALL IN THE FIRST PLACE MEAN SOME POSSIBLE ACTUAL, E.G. PHYSICAL, PROCESS IN THE REAL WORLD THAT HAS SOME RANDOM OR STOCHASTIC ELEMENT INVOLVED IN ITS STRUCTURE. IT WILL BE CONVENIENT, HOWEVER, ALSO TO USE THE SAME PHRASE FOR THE MATHEMATICAL REPRESENTATION AS WELL AS THE PHYSICAL CONCEPT, JUST AS WITH THE WORD 'PROBABILITY,' ESPECIALLY HERE WHERE WE SHALL BE MAINLY INTERESTED IN THE MATHEMATICAL THEORY IN ITS ROLE AS A THEORY OF STATISTICAL PHENOMENA.

" MANY OBVIOUS EXAMPLES OF SUCH PROCESSES ARE TO BE FOUND IN VARIOUS BRANCHES OF SCIENCE AND TECHNOLOGY, FOR EXAMPLE, THE PHENOMENON OF BROWNIAN MOTION, THE GROWTH OF A BACTERIAL COLONY, OR THE FLUCTUATING NUMBERS OF ELECTRONS AND PHOTONS IN A COSMIC-RAY SHOWER. IN MANY OF THESE EXAMPLES THE STATISTICAL OR RANDOM VARIABLES UNDER STUDY, SUCH AS THE COORDINATES OF A BROWNIAN PARTICLE, ARE CHANGING WITH TIME, BUT CHANGE INVOLVING ANY OTHER PARAMETER MAY ARISE; FOR EXAMPLE, A STOCHASTIC PROCESS INVOLVING SPACE PARAMETERS AS WELL AS TIME IS THE 'VELOCITY FIELD' OF A TURBULENT FLUID . . . .

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[14] BY PERMISSION FROM INTRODUCTION TO STOCHASTIC PROCESSES BY M. S. BARTLETT. COPYRIGHTED 1955. CAMBRIDGE UNIVERSITY PRESS.

"The mathematical theory which is the starting point of the theoretical developments is the theory of probability, as this is the basis of all statistical theory. In view of this central position of the mathematical theory of probability, its elements are summarized in the next section; but in view of the many controversial discussions over its interpretation , it may be as well to stress at once that we shall always use it as a theory about statistical phenomena. There are many situations where observations on particular phenomena can be repeated under similar conditions, but where, however closely one attempts to control the conditions under which the observations are made, there are irregular or random variations between the results of different trials. Nevertheless, a survey of all the trials often indicates regularities which stabilize as the number of trials is increased; such regularities are called statistical properties. (The word 'trials' is of course used here in a broad sense; thus in coin-tossing experiments we may consider either repeated tossing of the same coin or simultaneous tossing of many similar coins)."

The challenging scope of the probabilistic models are portrayed by Ulam in the following:[15] "Some of the great mathematicians of the eighteenth century, in particular Euler, succeeded in incorporating into the domain of mathematical analysis descriptions of many natural phenomena. Von Neumann's work attempted to cast in a similar role the mathematics stemming from set theory and modern algebra. This is of course, nowadays, a vastly more difficult undertaking. The infinitesimal calculus and the subsequent growth of analysis through most of the nineteenth century led to hopes of not merely cataloguing, but of understanding the contents of the Pandora's box opened by the discoveries of physical sciences. Such hopes are now illusory, if only because the real number system of the Euclidean space can no longer claim, algebraically, or even only topologically, to be the unique or even the best mathematical substratum for physical theories. The physical ideas of the 19th century, dominated mathematically by differential and integral equations and the theory of analytic functions, have become inadequate. The new quantum theory requires on the analytic side a set-theoretically more general point of view, the primitive notions themselves involving probability distributions and infinite-dimensional function spaces. The algebraical counterpart to this involves a study of combinatorial and algebraic structures more general than those presented by real or complex numbers alone.

"Another major source from which general mathematical investigations are beginning to develop is a new kind of combinatorial analysis stimulated by the recent fundamental researches in the biological sciences. Here, the lack of general method at the present time is even more noticeable. The problems are essentially non-linear, and of an extremely complex combinatorial character; it seems that many years of experimentation and heuristic studies will

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[15] By permission from John von Neumann 1903-1957 by S. Ulam. Copyrighted 1958. American Mathematical Society Vol. 64, Nr 3, Part 2, May 1958.

be necessary before one can hope to achieve the insight required for decisive syntheses. An awareness of this is what prompted von Neumann to devote so much of his work of the last ten years to the study and the construction of computing machines and to formulate a preliminary outline for the study of automata."

Wiener has the following to say with regard to the role of statistics in physics:[16] " . . . Newtonian physics, which had ruled from the end of the seventeenth century to the end of the nineteenth with scarcely an opposing voice, described a universe in which everything happened precisely according to law, a compact, tightly organized universe in which the whole future depends strictly upon the whole past. Such a picture can never be either fully justified or fully rejected experimentally and belongs in large measure to a conception of the world which is supplementary to experiment but in some ways more universal than anything that can be experimentally verified. We can never test by our imperfect experiments whether one set of physical laws or another can be verified down to the last decimal. The Newtonian view, however, was compelled to state and formulate physics as if it were, in fact, subject to such laws. This is now no longer the dominating attitude of physics, and the men who contributed most to its downfall were Bolzmann in Germany and Gibbs in the United States.

"These two physicists undertook a radical application of an exciting, new idea. Perhaps the use of statistics in physics which, in large measure, they introduced was not completely new, for Maxwell and others had considered worlds of very large numbers of particles which necessarily had to be treated statistically. But what Bolzmann and Gibbs did was to introduce statistics into physics in a much more thorough going way, so that the statistical approach was valid not merely for systems of enormous complexity, but even for systems as simple as the single particle in a field of force.

" . . . There was, actually, an important statistical reservation implicit in Newton's work, though the eighteenth century, which lived by Newton, ignored it. No physical measurements are ever precise; and what we have to say about a machine or other dynamic system really concerns not what we must expect when the initial positions and moments are given with perfect accuracy (which never occurs), but what we are to expect when they are given with attainable accuracy. This merely means that we know, not the complete initial conditions, but something about their distribution. The functional part of physics, in other words, cannot escape considering uncertainty and the contingency of events. It was the merit of Gibbs to show for the first time a clean-cut scientific method for taking this contingency into consideration . . . .

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[16] By permission from The Human Use of Human Beings by N. Wiener. Copyrighted . Houghton Mifflin Co.

"Gibbs had to work with theories of measure and probability which were already at least twenty-five years old and were grossly inadequate to his needs. At the same time, however, Borel and Lebesque in Paris were devising the theory of integration which was to prove opposite to the Gibbsian idea. Borel was a mathematician who had already made his reputation in the theory of probability and had an excellent physical sense. He did work leading to this theory of measure, but he did not reach the stage in which he could close it into a complete theory. This was done by his pupil Lebesque, who was a very different sort of person. He had neither the sense of physics nor an interest in it. Nonetheless Lebesque solved the problem put by Borel, but he regarded the solution of this problem as more than a tool for Fourier series and other branches of pure mathematics. A quarrel developed between the two men when they both became candidates for admission to the French Academy of Sciences, and only after a great deal of mutual denigration, did they both receive this honor. Borel, however, continued to maintain the importance of Lebesque's work and his own as a physical tool; but I believe that I myself, in 1920, was the first person to apply the Lebesque integral to a specific physical problem - that of the Brownian motion.

"This occurred long after Gibb's death, and his work remained for two decades one of those mysteries of science which work even though it seems that they ought not to work. Many men have had intuitions well ahead of their time; and this is not least true in mathematical physics. Gibb's introduction of probability into physics occurred well before there was an adequate theory of the sort of probability he needed. But for all these gaps it is, I am convinced, Gibbs rather than Einstein or Heisenberg or Planck to whom we must attribute the first great revolution of twentieth century physics.

"This revolution has had the effect that physics now no longer claims to deal with what will always happen, but rather with what will happen with an overwhelming probability. At the beginning in Gibb's own work this contingent attitude was superimposed on a Newtonian base in which the elements whose probability was to be discussed were systems obeying all of the Newtonian laws. Gibb's theory was essentially new, but the permutations with which it was compatible were the same as those contemplated by Newton. What has happened to physics since is that the rigid Newtonian basis has been discarded or modified, and the Gibbsian contingency now stands in its complete nakedness as the full basis of physics. It is true that the books are not yet quite closed on this issue and that Einstein and, in some of his phases, DeBroglie, still contend that a rigid deterministic world is more acceptable than a contingent one; but these great scientists are fighting a rear-guard action against the overwhelming force of a younger generation.

" . . . The important thing is that in Einstein's work, light and matter are on an equal basis, as they have been in the writings before Newton; without the Newtonian subordination of everything else to matter and mechanics . . . .

" . . . In explaining his views, Einstein makes abundant use of the observer who may be at rest or may be moving. In his theory of relativity it is impossible to introduce the observer without also introducing the idea of message, and without, in fact, returning the emphasis of physics to a quasi-Leibnitizian state, whose tendency is once again optical. Einstein's theory of relativity and Gibb's statistical mechanics are in sharp contrast, in that Einstein, like Newton, is still talking primarily in terms of an absolutely rigid dynamics not introducing the idea of probability."

Lurie has the following to say regarding the role of the statistician and scientists:[17] " . . . The statistician, if he is really to assist the scientist, must perform a necessary, but irritatingly annoying task; he must ask the scientist impertinent questions. Indeed, the questions, if bluntly asked, may appear to be not only impertinent but almost indecently prying -- because they deal with the foundations of the scientist's thinking. By these questions unsuspected weaknesses in the foundations may be brought to light, and the exposure of weaknesses in one's thinking is a rather unpleasant occurrence.

"The statistician will, then, if he is wise in the ways of human beings as well as learned in statistics, ask these questions diplomatically or even not ask them as questions at all. He may well guide the discussion with the scientist in such a way that the answers to the questions will be forthcoming without the questions having been even explicitly asked.

"And if happily the scientific and statistical disciplines reside within one mind, and it is the scientist's statistical conscience which asks him these questions, instead of impertinent questioning there is valid scientific soul-searching.

"Regardless, then, of whether these questions arise inside or outside the scientist's own mind, what are they? These:

1. With respect to the experiment you are performing, just what are your ideas?

2. With respect to the scientific area to which these ideas refer, just what are they about?

3. How sure do you want to be of the correctness of these ideas?

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[17] By permission from <u>The Impertinent Questioner</u> by W. Lurie. Copyrighted 1958. <u>American Scientist Vol. 46</u>, Nr 1, March 1958.

"IN ORDER TO UNDERSTAND THE STATISTICIAN'S REASONS FOR ASKING THESE QUESTIONS, LET US FIRST SEE HOW THE SCIENTIST'S ACTIVITIES LOOK TO THE STATISTICIAN.

"FROM THE STATISTICIAN'S POINT OF VIEW, WHAT THE SCIENTIST DOES, IS: PERFORM EXPERIMENTS AND/OR MAKES OBSERVATIONS TO OBTAIN DATA RELATING TO AN IDEA HE HAS ABOUT THE ORGANIZATION OF THAT PORTION OF THE WORLD HE IS INTERESTED IN, SO THAT HE CAN DECIDE WHETHER HIS IDEA WAS CORRECT OR NOT.

"FOR EACH OF THESE UNDERLINED ASPECTS OF THE SCIENTIST'S ACTIVITY, THERE IS A CORRESPONDING QUESTION.

AN IDEA HE HAS

" . . . THE IMPERTINENT QUESTIONER MUST TAKE THE RISK OF APPEARING TO IMPLY THAT THE SCIENTIST IS NOT THINKING CLEARLY. AND, OF COURSE, EVEN AN IMPLICATION TO THIS EFFECT IS NOT CALCULATED TO ENDEAR THE IMPLIER TO THE HEART OF THE IMPLYEE. BUT IT IS EXACTLY THIS IMPLICATION THAT, PERHAPS INNOCENTLY, IS ASSOCIATED WITH THE QUESTION, 'JUST WHAT ARE YOUR IDEAS.'

" WHY DOES THE STATISTICIAN ASK THIS IMPERTINENT QUESTION? BECAUSE IT IS A PRECONDITION FOR THE STATISTICIAN'S BEING ABLE TO HELP THE SCIENTIST ACCOMPLISH HIS OBJECTIVE. A HAZILY FORMULATED IDEA NOT ONLY CAN BE DISCUSSED, AT BEST, WITH DIFFICULTY, BUT FURTHER, IT IS PRACTICALLY IMPOSSIBLE TO TEST ITS CORRECTNESS. THEREFORE, THE STATISTICIAN HAS A RULE, HIS NAME FOR WHICH IS: EXPLICIT HYPOTHESIZATION. THIS RULE EXPRESSES THE REQUIREMENT THAT THE IDEA, WHOSE CORRECTNESS IS TO BE DETERMINED BY THE EXPERIMENT, SHOULD BE STATED IN AS CLEAR, DETAILED AND EXPLICIT FORM AS POSSIBLE, PREFERABLY BEFORE THE EXPERIMENT IS CONDUCTED. THIS IDEA CAN RELATE EITHER TO THE INFLUENCE OF ONE FACTOR OR TO THE INFLUENCE OF SEVERAL FACTORS, OR TO THE NUMERICAL CHARACTERIZATION OF A PROPERTY (OR PROPERTIES) OF WHATEVER IS BEING EXPERIMENTED ON. IN THE EARLY STAGES OF AN INVESTIGATION, WHERE WHAT ARE BEING SOUGHT ARE THE INFLUENTIAL FACTORS (I.E. THOSE WHICH, WHEN THEY ARE AT VARYING LEVELS, GIVE RISE TO SUFFICIENTLY VARIED RESULTS) THE IDEA (OR HYPOTHESIS) NEED NOT BE SPECIFIC, BUT IT MUST BE EXPLICIT. THE HYPOTHESIS CAN BE BROAD, BUT IT MUST BE EXPLICITLY BROAD: - THAT IS, EVEN THOUGH IT IS NOT A HYPOTHESIS ABOUT DETAILS, ITS BOUNDARY MUST BE SHARPLY DELINEATED . . . .

"NOW ASSUMING THAT THE HYPOTHESIS HAS BEEN SUFFICIENTLY EXPLICITLY FORMULATED, THE SCIENTIST AND STATISTICIAN CAN TOGETHER REVIEW THE PLAN (OR DESIGN) OF THE EXPERIMENT, AND ASSURE THEMSELVES THAT SUCH DATA WILL BE OBTAINED AS WILL BE SUFFICIENT TO DETERMINE THE CORRECTNESS (OR NON-CORRECTNESS) OF THE SCIENTIST'S IDEA.

THAT PORTION OF THE WORLD HE IS INTERESTED IN

" . . . AGAIN, THE IMPERTIENT QUESTIONER MUST BE CAREFUL IN ASKING: 'JUST WHAT ARE YOUR IDEAS ABOUT?' EVEN THOUGH ONE MAY ADMIT THAT HIS IDEAS are not as clearly and explicitly formulated as he would like, the question 'Just what are your ideas about?' carries with it, to the person being asked, the implication that he isn't clear about the subject-matter of his ideas, surely not a flattering implication. The statistician has a reason for his implied aspersion on the basis of the scientist's self-esteem. The statistician's reason can be stated to the scientist thus 'It's for your own good. If I am to help you decide, on the basis of the experimental facts, whether your ideas are correct or not, I have to know, as explicitly as possible, not only what your ideas are, but what they are about. My name for this requirements is: Model Formulation.' Technically, Model Formulation establishes the requirement that a clear differentiation be made as to whether the scientist's ideas are intended to be applicable only to the conditions of the experiment (the narrower range of application) or to conditions (i.e., levels of the factors) other than those specific ones under which the experiment is being conducted (the broader range of application). Why the necessity for this differentiation? Because, when the experimental data have been obtained, the analysis of the data is carried on in different ways, depending on whether the hypotheses are intended to have the broader or narrower range of application.

Whether His Idea Was Correct Or Not

" . . . The statistician's third question -- 'How sure do you want to be of the correctness of your ideas?' is the least impertinent of the three. This question, unlike the other two, does not probe the foundations of the scientist's thinking, but rather requests him to quantify a previously unquantified aspect of it. (In fact, the request is in accordance with the scientist's own predilection for quantitative data). This aspect is that dealing with levels of assurance, for which ordinary language supplies us with qualitatively descriptive terms (somewhat sure, rather sure, quite sure, extremely sure). But these terms are not sufficiently explicit for scientific use. Therefore, the statistician asks the scientist to decide upon and express his desired level of assurance in quantitative terms, so that it can be determined, by analysis of the quantitative data, whether the desired level of assurance of the conclusions has been achieved. The statistician's name for the choice and quantitative expression of the desired level of assurance is: Significance Level Selection. And how does the statistician help the scientist choose the desired level of assurance? By bringing to the forefront of the scientist's consciousness his already unconscious awareness of the inherent variability of events (i.e., that, because of chance alone, no repetition of an experiment will give exactly the same results); by helping the scientist decide what assurance is desired that the hypothesis has not been 'confirmed' just by the operation of chance alone; and by furnishing the mathematical tools to decide, on the basis of the experimental data, whether the desired level of assurance has been attained.

" . . . When the scientist has selected the chance he is willing to take of being wrong (or what is equivalent, how sure he wants to be that he is correct) in his conclusions, the statistician can analyze the data and tell the scientist what conclusions he can validly draw (i.e. what decisions

HE CAN MAKE ABOUT THE CORRECTNESS OF HIS IDEAS).

EPILOGUE

" . . . ONE FINAL WORD. IT IS THE STATISTICIAN'S RESPONSIBILITY TO ASK THESE QUESTIONS, NOT TO ANSWER THEM. IT IS THE SCIENTIST'S RESPONSIBILITY TO DECIDE EXACTLY WHAT HIS HYPOTHESES ARE, WHAT THESE HYPOTHESES ARE ABOUT, AND HOW SURE HE WANTS TO BE OF THEIR CORRECTNESS.

"THE STATISTICIAN, IN ASKING HIS IMPERTINENT QUESTIONS, IS JUST EXPLICITLY BRINGING TO THE SCIENTIST'S ATTENTION, RESPONSIBILITIES THAT THE SCIENTIST MAY NOT HAVE BEEN AWARE THAT HE HAD. AND THE MORE THE SCIENTIST BECOMES AWARE OF HIS RESPONSIBILITIES, AND TAKES THEM INTO ACCOUNT IN HIS WORK, SO MUCH MORE ACCURATE AND VALID WILL HIS CONCLUSIONS BE, AND SO MUCH MORE PROPERLY RELATED TO THE REALITY WITH WHICH HE DEALS."

PROFESSOR NOYES OF THE UNIVERSITY OF CALIFORNIA HAS THE FOLLOWING TO SAY WITH RESPECT TO MATHEMATICAL MODELS OF THEORETICAL PHYSICS:[18] "THUS I HAVE A STRONG FEELING THAT FROM NOW ON THE MATHEMATICAL MODELS OF THEORETICAL PHYSICS ARE UNLIKELY TO BE UNIQUE, AND THE ADOPTION OF ONE RATHER THAN ANOTHER WILL COME TO BE BASED MORE ON AESTHETIC CONSIDERATIONS THAN ON EXPERIMENTAL CRITERIA. ONCE THIS SITUATION ACTUALLY EXISTS (RATHER THAN POTENTIALLY AS IS TRUE AT PRESENT) IT WOULD APPEAR TO ME OBVIOUS THAT THE SUCCESS OF PHYSICS WOULD HAVE NOTHING TO DO WITH THE EXISTENCE OF A HYPOTHETICAL 'REAL' WORLD.

"BUT THE DEMONSTRATION OF THE NON-UNIQUENESS OF SUCCESSFUL PHYSICAL THEORIES HAS YET TO BE CARRIED THROUGH IN A NON-TRIVIAL CASE (IT IS CLEARLY ALWAYS POSSIBLE TO MODIFY AN EXISTING THEORY BY ADDING STRUCTURES WHICH HAVE A DIFFERENT INTUITIVE PHYSICAL INTERPRETATION, BUT WHICH ARE ADDED IN SUCH A WAY AS TO HAVE NO OBSERVATIONAL CONSEQUENCES). I THEREFORE WILL PRESENT ANOTHER ARGUMENT, DRAWN FROM THE MATHEMATICAL BASIS OF PHYSICAL THEORY, WHICH LEADS TO THE SAME CONLCUSION. UP TO THIS POINT I HAVE SPOKEN OF THE MATHEMATICAL CONCLUSIONS OF OUR THEORY AS IF THEY WERE UNIQUE, UNAMBIGUOUS, AND FLOWING ANALYTICALLY FROM OUR PREMISES, ALTHOUGH I HAD TO DO SOME FAST TALKING WHEN IT CAME TO THE REMOVAL OF INFINITIES FROM OUR THEORY. BUT, ALTHOUGH THERE ARE THREE DISTINCT SCHOOLS OF THOUGHT ABOUT THE FOUNDATIONS OF MATHEMATICS, I BELIEVE THEY WOULD UNITE IN DENYING THAT THIS IS TRUE.

"THE INTUITIONISTS HAVE HELD FOR A LONG TIME THAT A LARGE PORTION OF MODERN MATHEMATICS IS SHOCKINGLY DEFICIENT IN THE TYPE OF RIGOR THEY REQUIRE A MATHEMATICAL THEORY TO EXHIBIT, AND THEIR BAN WOULD CERTAINLY FALL ON ALMOST EVERY MATHEMATICAL INGREDIENT OF THE THEORIES I HAVE BEEN DISCUSSING. FROM THEIR POINT OF VIEW, FEW OF THE ABOVE CONCLUSIONS HAVE ANY MATHEMATICAL JUSTIFICATION WHATSOEVER. THE MATHEMATICAL LOGICIANS HOLD THAT MATHEMATICS

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[18] BY PERMISSION FROM THE PHYSICAL DESCRIPTION OF ELEMENTARY PARTICLES BY P. H. NOYES. COPYRIGHTED 1957. THE AMERICAN SCIENTIST, VOL. 45, NR 5, DECEMBER 1957.

IS A BRANCH OF LOGIC. BUT LOGIC IS BY NO MEANS UNIQUE, AS THE CREATION OF MULTI-VALUED LOGICS HAS SHOWN. CONSEQUENTLY, FROM THIS POINT OF VIEW, EACH LOGIC GENERATES ITS OWN MATHEMATICS, AND THE MATHEMATICAL STRUCTURE WE HAVE BEEN USING COULD PRESUMABLY BE REPLACED BY ONE BASED ON A DIFFERENT LOGIC AND LEADING TO A DIFFERENT PHYSICAL PICTURE OF THE UNDERLYING STRUCTURE. FINALLY, THE FORMALIST GOEDEL HAS PROVED THAT ANY MATHEMATICAL STRUCTURE RICH ENOUGH TO GENERATE THE NATURAL NUMBERS (WHICH IS CERTAINLY TRUE OF THE MATHEMATICS USED IN PHYSICS) CONTAINS UNDECIDABLE PROPOSITIONS, WHICH CAN NEITHER BE PROVED TRUE NOR FALSE WITHIN THE SYSTEM. CONSEQUENTLY MATHEMATICAL PHYSICS MUST ALSO CONTAIN UNDECIDABLE PROPOSITIONS ABOUT THE PHYSICAL STRUCTURES IT DESCRIBES. THEREFORE, FROM THESE THREE BASIC POINTS OF VIEW, A MATHEMATICAL DESCRIPTION OF A PHYSICAL SYSTEM MUST CONTAIN ELEMENTS WHICH EVEN MATHEMATICALLY SPEAKING ARE ARBITRARY, OR AMBIGUOUS OR UNDECIDABLE.

"FOR ME, AT LEAST, THE CONCLUSION IS INESCAPABLE THAT THE STRUCTURE OF MATHEMATICAL PHYSICS CAN TELL US LITERALLY NOTHING ABOUT A HYPOTHETICAL 'REAL' WORLD. IN ONE SENSE ONLY CAN THE STRUCTURE BE SAID TO BE UNIQUE, THAT IS, IT MUST GIVE AN UNAMBIGUOUS DESCRIPTION OF OUR IMMEDIATELY GIVEN SENSE IMPRESSIONS, AND HOWEVER FAR WE GO UP THE ABSTRACTION LADDER FROM THIS BASIS, OUR THEORY MUST ALLOW US TO RETRACE OUR STEPS AND REPRODUCE THE STRUCTURE OF THESE EXPERIENCES, IF IT IS TO BE CONSIDERED SUCCESSFUL. BUT SINCE, MATHEMATICALLY SPEAKING, OUR CONCLUSIONS DO NOT FOLLOW ANALYTICALLY FROM OUR PREMISES, IT SEEMS TO ME OBVIOUS THAT THE RESULTING ABSTRACT STRUCTURE CAN NEVER BE UNIQUE. THIS FACT ALONE SEEMS SUFFICIENT GUARANTEE THAT THE APPROACH VIA MATHEMATICAL PHYSICS CAN NEVER DO MORE THAN DESCRIBE THE IMMEDIATELY GIVEN STRUCTURE OF EXPERIENCE IN TERMS OF SYMBOLS WHOSE SELECTION IS ESSENTIALLY ARBITRARY, AND WHICH HAVE NO ULTIMATE SIGNIFICANCE IN THEMSELVES, HOWEVER SUCCESSFUL THE PHYSICAL THEORY THEY DEPICT.

"FINALLY I WOULD JUST LIKE TO MENTION AGAIN THE METHODOLOGICAL POINT WITH WHICH WE STARTED. I HOPE AS A MINIMUM ACCOMPLISHMENT OF THIS PAPER TO HAVE CONVEYED THE IMPRESSION THAT IN AT LEAST ONE BRANCH OF PHYSICS MATHEMATICAL STRUCTURE HAS REPLACED 'INFERRED ENTITIES' AS A FRUITFUL WORKING TOOL OF THE PHYSICIST."

SIMULATION

THE ENGINEERING CAPABILITY OF MATHEMATICALLY DESCRIBING INCREASINGLY COMPLEX INTEGRATED WEAPON SYSTEMS IS CONTINUALLY PYRAMIDING. THE ELECTRONIC COMPUTERS WHICH ARE BEING DESIGNED TO HANDLE THESE PROBLEMS CONTINUES TO LAG BY ONE OR TWO ORDERS OF MAGNITUDE BEHIND. FOR EXAMPLE, A DECADE AGO, THE TYPICAL COMPUTER FACILITY COULD HANDLE AUTOPILOT STUDIES WITH AN EXTREMELY SIMPLIFIED AIRFRAME. THE PRESENT TYPICAL COMPUTER FACILITY IS APPROACHING THE CAPACITY TO SIMULTANEOUSLY SIMULATE THE GUIDANCE, AUTOPILOT AND SIX DEGREE OF FREEDOM AIRFRAME COMBINATION (FLAT NONROTATING EARTH). TOMORROWS TYPICAL COMPUTING FACILITY WILL HAVE THE CAPABILITY TO INCLUDE WITH THE AIRFRAME, AUTOPILOT AND GUIDANCE SUCH THINGS AS AIRFRAME FLEXURE, MOTOR INTERIOR BALLISTICS, MISSILE LAUNCH CONTROL LOGIC, MULTIPLE TARGETS, COUNTERMEASURES, ETC., OVER A ROTATING SPHEROIDAL EARTH.

As Bauer points out, the separate effects of any one of the above areas can be studied adequately on small simulators; however the combined effects and the mutual interactions of these phenomena can be analyzed only on a large simulator.

Irrespective of the various current opinions pro and con on large scale simulations, the users of simulation facilities have a tendency to include more and more of the complications in the formulation of the complete problem.

Since there are many important classes of problems which cannot even be stated mathematically, one must resort also to physical simulation. When one considers the distinction between computing, simulation, analog, digital, etc., sharp lines of demarcation are not realistic; even though some people argue as though they are. Rather than being concerned about definitions an attempt will be made to point out the most important aspect of system simulation.

Consider a weapon system composed of both digital and analog devices as shown in Fig. 1-1. A distinction between a physical system and a mathematical system (very loosely used) is made. Thus, the mathematical model of the physical system may be written in terms of functions of continuous variables or in terms of functions of discrete variables. Once the mathematical system of equations is mechanized on a computer, one has built a physical system model of the original weapon system. The model physical system in most cases is an electronic model or electro-mechanical depending on the types of computing elements one is using. Thus a digital computer model of the weapon system digital devices may be said to be the "analog" of that portion of the weapon system.

The gist of this general discussion is: Large weapon system simulations on general purpose computers can most effectively be achieved by a team having knowledge of the physical system as well as knowledge of the computer in general. Furthermore, the operations of the simulation computers must be approached with built in subsystem checks on the physical system behavior as well as an understanding of the "black-box" computing elements.

Vjada [19] states that, "in most applications of linear programming or game theory that the computations called for are so extensive that it would seem impossible to carry them out in the lifetime of those who are interested in the answer and that computing methods and the advent of automatic computers has made it possible to hope for answers within a reasonable time. This book will deal with the computational aspect rather thoroughly . . . ."

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[19] By permission from The Theory of Games and Linear Programming by S. Vjada. Copyrighted . Methuen and Company, Ltd.

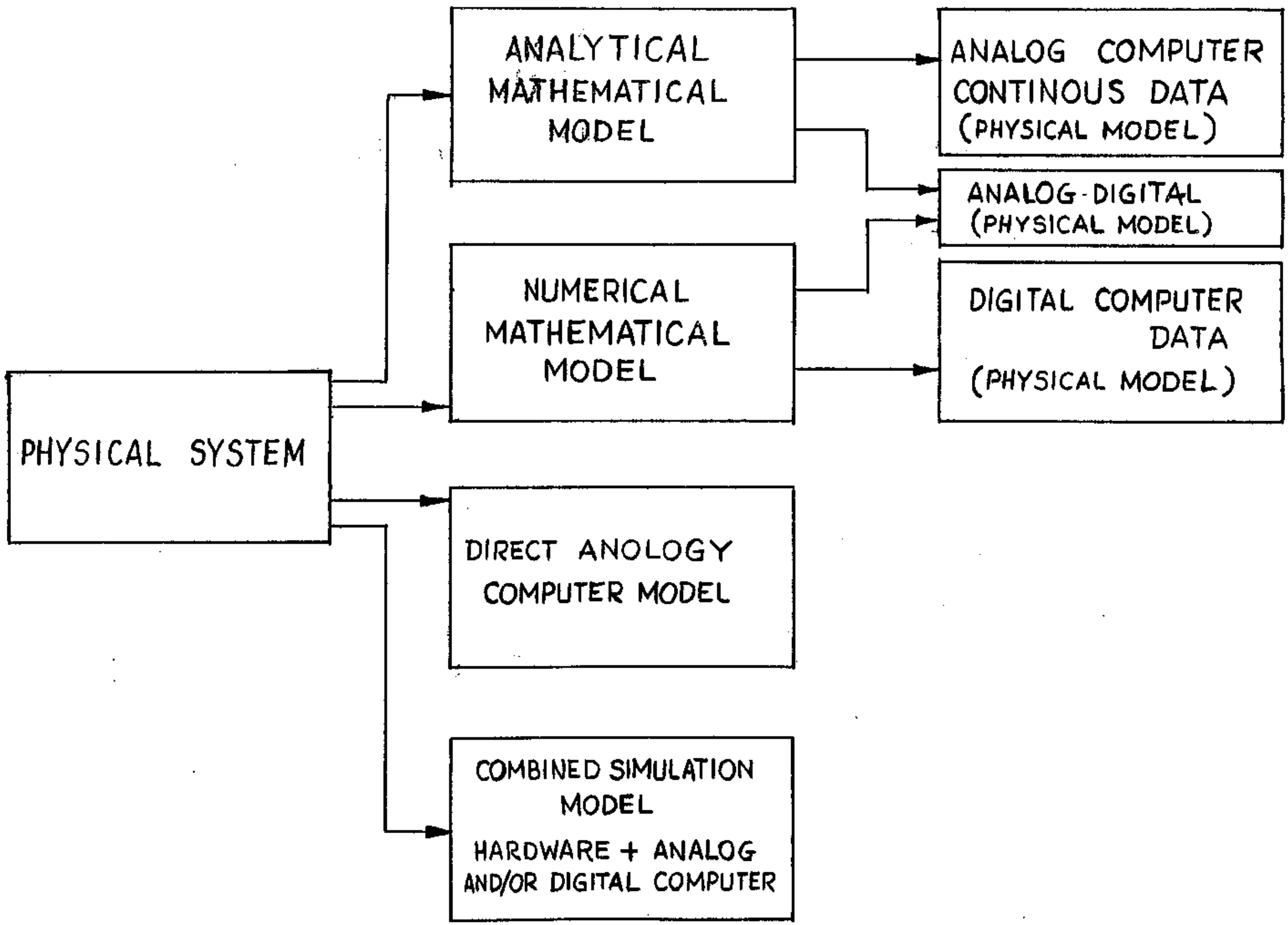

Fig. 1-1 Physical And Mathematical Systems

Reese[20] says, "In computing, the availability of large fast machines has changed the face of industrial research, and we shall be hard pressed to provide trained users of these machines. As the telephone company's leased lines problem suggests, one of the boundary conditions imposed on many industrial problems is the ease with which the proposed formulation or solution yields to evaluation by automatic computation. Moreover the availability of large-scale fast computers has multiplied the number of problems for which industry seeks mathematical solution, and has given industry a powerful tool for attacking some problems for which no analytical procedures were formerly sought."

With regards to analog and digital devices Lehmer states:[21] "Every engineer knows that computing devices are of two types, analog and digital (to employ the terms commonly used in this country), and he knows well the distinctive features of each. The author has come to the conclusion, in his conversations with engineers about their preferences in computing devices, that they favor the analogue device for actual use but prefer to build a digital device. This is quite natural. The engineer has been trained to think of a function as a smooth curve, and he is certainly more at home with recording instruments such as voltmeters and oscilloscopes than with counting mechanisms. Also the engineer with a problem has been taught to use small scale models, so that a simulator or an equivalent circuit certainly appeals to him as more 'down to earth' than the really down-to-earth method of constructing a mathematical model and solving the mathematical problem the procedure necessary for the application of digital techniques.

"When it comes to construction, the digital computer certainly offers a wider variety of engineering experience. Until recently, the main effort in improving the analogue device has been in the direction of promoting the manufacture of more accurate and constant components, a necessary but rather uninspiring task. Now there is a growing tendency to attach new types of equipment to the analogue installation in order to increase its flexibility and over-all usefulness. This tendency, if continued, will attract more engineers to this field than in the past . . . .

"Perhaps the most significant limitation of the analogue device is not its low accuracy but its lack of logical flexibility. The fact is that it handles only one independent variable, time. The outstanding problems of

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20 By permission from Mathematicians In The Marketplace by N. Rees. Copyrighted 1958. American Mathematical Monthly, Vol. 65, Nr 5, May 1958.

21 By permission from Mathematics For The Modern Engineer by E. F. Beckenbach. Copyrighted 1956. McGraw-Hill Book Company, Inc.

TODAY ARE NO LONER PROBLEMS CONCERNING ORDINARY DIFFERENTIAL EQUATIONS, BUT ARE PROBLEMS OF SYSTEMS WHICH ARE DISTRIBUTED IN SPACE AND LEAD TO PARTIAL DIFFERENTIAL EQUATIONS IN TWO OR MORE INDEPENDENT VARIABLES. IT IS TRUE THAT, WITH SOME MATHEMATICAL KNOW-HOW CERTAIN OF THESE PARTIAL DIFFERENTIAL EQUATIONS CAN BE 'SEPARATED' INTO ORDINARY DIFFERENTIAL EQUATIONS WITH THE BOUNDARY CONDITIONS GIVING RISE TO EIGENVALUE PROBLEMS. IN MANY CASES, HOWEVER, THIS TECHNIQUE IS NOT APPLICABLE. OF COURSE, DISTRIBUTED SYSTEMS CAN SOMETIMES BE 'LUMPED' TO PRODUCE AN ORDINARY DIFFERENTIAL EQUATION OF AN EQUIVALENT PROBLEM BUT THIS TAKES REAL UNDERSTANDING OF THE PROBLEM IN ORDER TO GIVE MORE THAN A QUALITATIVE INTERPRETATION OF THE WORD 'EQUIVALENT . . . .'

THE TERM 'DISCRETE-VARIABLE DEVICE,' THOUGH NOT IN SUCH COMMON USE, REALLY DESCRIBES THE ESSENTIAL FEATURE OF THE DIGITAL COMPUTER. FROM THE POINT OF VIEW OF THE DISCRETE-VARIABLE DEVICE, THINGS ARE TO BE COUNTED RATHER THAN MEASURED; MATHEMATICS IS NOT GEOMETRY BUT ARITHMETIC; THE UNIVERSE IS QUANTIZED, AND THIS INCLUDES MATHEMATICS. INTEGRALS ARE BUT SUMS, AND DERIVATIVES ARE BUT DIFFERENCE QUOTIENTS; FUNCTIONS ARE DISCONTINUOUS EVERYWHERE; LIMITS, INFINITIES, AND INFINITESIMALS DO NOT REALLY EXIST. WORSE THAN THAT, DIVISION IS NOT THE INVERSE OF MULTIPLICATION, AND EVEN ADDITION IS NOT ALWAYS POSSIBLE (BECAUSE OF OVERFLOW) . . . .

"A FEW WORDS MAY BE SAID ABOUT A FINAL CLASS OF PROBLEMS, THOSE INVOLVING DISCRETE VARIABLES. MANY OF THESE ARE RELATIVELY NEW PROBLEMS AND OWE THEIR INTEREST AND IMPORTANCE TO THE RISE OF THE DIGITAL COMPUTER, FOR WITHOUT SUCH INSTRUMENTS OF CALCULATION MOST OF THESE PROBLEMS WOULD BE IMPOSSIBLE TO SOLVE.

"DISCRETE-VARIABLE PROBLEMS ARISE IN SUCH FIELDS AS OPERATIONS ANALYSIS, LOGISTICS, CRYSTAL AND MOLECULAR STRUCTURE, AND THE DESIGN OF EXPERIMENTS. THEY ALSO ARISE IN THE DESIGN AND PROGRAMMING OF DIGITAL COMPUTERS. IN ALL SUCH PROBLEMS WE HAVE A FINITE BUT OFTEN LARGE NUMBER OF CHOICES OF VARIABLES HAVING FINITE DIFFERENCES. ANY SUCH CHOICE RESULTS IN A SITUATION TO WHICH WE CAN ATTACH A MEASURE OF GOODNESS.

"THUS IN ADOPTING THE DISCRETE-VARIABLE POINT OF VIEW WE SEEM TO GO BACK TO PYTHAGORAS. BEFORE BECOMING UNDULY ALARMED AT SUCH A PROSPECT, THE ENGINEER SHOULD REALIZE THAT HE IS ALREADY WELL VERSED IN DISCRETE-VARIABLE TECHNIQUES IN HIS EVERYDAY USE OF THE DESK CALENDAR. IN FACT, THE TYPICAL DIGITAL COMPUTER DOES THE SAME THINGS THAT A DESK CALCULATOR AND ITS OPERATOR DO, ONLY IT DOES THEM AUTOMATICALLY AND RAPIDLY."

JOHN HARLING OF ORBIT LTD, LONDON, ENGLAND CLAIMS:[22] "CONSIDERABLE CONFUSION EXISTS OVER THE BEST TERMINOLOGY TO USE. THE TERM 'MONTE CARLO' IS PRESENTLY SOMEWHAT FASHIONABLE; THE TERM 'SIMULATION' IS TO BE PREFERRED,

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[22] BY PERMISSION FROM SIMULATIONS TECHNIQUE IN OPERATIONS RESEARCH-A REVIEW BY J. HARLING. COPYRIGHTED 1954. THE JOURNAL OF OPERATIONS RESEARCH SOCIETY OF AMERICA, VOL. 65, NR 3, MAY-JUNE 1958.

BECAUSE IT DOES NOT SUGGEST THAT THE TECHNIQUE IS LIMITED TO WHAT IS FAMILIAR TO STATISTICIANS AS A SAMPLING EXPERIMENT.

"BY SIMULATION IS MEANT THE TECHNIQUE OF SETTING UP A STOCHASTIC MODEL OF A REAL SITUATION, AND THEN PERFORMING SAMPLING EXPERIMENTS UPON THE MODEL. THE FEATURE WHICH DISTINGUISHES A SIMULATION FROM A MERE SAMPLING EXPERIMENT IN THE CLASSICAL SENSE IS THAT OF THE STOCHASTIC MODEL. WHEREAS A CLASSICAL SAMPLING EXPERIMENT IN STATISTICS IS MOST OFTEN PERFORMED DIRECTLY UPON RAW DATA, A SIMULATION ENTAILS FIRST OF ALL THE CONSTRUCTION OF AN ABSTRACT MODEL OF THE SYSTEM TO BE STUDIED.

"IN USING A SIMULATION TO INVESTIGATE THE PROPERTIES OF A REAL SYSTEM, THE FOLLOWING STAGES MAY BE DISTINGUISHED:

1. REDUCTION OF THE RAW DATA OF THE PROBLEM TO AN APPROPRIATE FORM.

2. CONSTRUCTION OF A MODEL OF THE REAL SYSTEM WHICH NEITHER OVER-SIMPLIFIES THE SYSTEM TO THE POINT WHERE THE MODEL BECOMES TRIVIAL, NOR CARRIES OVER SO MANY FEATURES FROM THE REAL SYSTEM THAT THE MODEL BECOMES INTRACTABLE AND PROHIBITIVELY CLUMSY.

3. BRINGING THE DATA AND THE MODEL TOGETHER IN A SAMPLING EXPERIMENT, WHOSE PURPOSE IS TO DISCOVER HOW THE REAL SYSTEM BEHAVES UNDER A VARIETY OF PRESCRIBED CONDITIONS.

"AT EACH OF THESE STAGES, THERE ARE SEVERAL PRECAUTIONS TO BE TAKEN, AND MANY USEFUL TECHNIQUES EXIST FOR EXTRACTING THE MAXIMUM AMOUNT OF INFORMATION FROM THE DATA AVAILABLE IN A REASONABLE PERIOD OF TIME AND AT A REASONABLE COST. THESE TECHNIQUES WILL BE DISCUSSED BRIEFLY IN THE PAPER.

"OWING PARTLY TO THE CONFUSION OF TERMINOLOGY, AND PARTLY TO THE FACT THAT SPECIALISTS IN MANY DIFFERENT FIELDS HAVE WORKED ON THE SIMULATION PROBLEM, THERE IS LITTLE PUBLISHED LITERATURE ON SIMULATION, AS COMPARED WITH MONTE CARLO TECHNIQUES. MOST WRITERS HAVE APPROACHED THE PROBLEM EITHER AS PURE MATHEMATICIANS WHO USE A SAMPLING EXPERIMENT TO SOLVE A DETERMINISTIC PROBLEM, OR AS STATISTICIANS WHO PROPERLY ENOUGH HAVE BEEN CONCERNED WITH REDUCING THE VARIANCE OF THEIR SAMPLING EXPERIMENTS. LITTLE HAS BEEN WRITTEN ABOUT THE OPERATIONS RESEARCH USE OF SIMULATION, AND SUCH PRACTICAL APPLICATIONS AS HAVE BEEN PUBLISHED SEEM TO MAKE LITTLE USE OF THE VERY VALUABLE WORK THAT THE THEORETICIANS HAVE DONE. THE TENDENCY SEEMS TO HAVE BEEN EITHER TO PERFORM 'BRUTE FORCE' SIMULATIONS LACKING ANY SORT OF STOCHASTIC INGENUITY, OR ELSE TO BE EXCLUSIVELY PREOCCUPIED WITH THE FORMAL PROBLEMS INVOLVED . . . .

"NEARLY ALL OPERATIONS-RESEARCH SIMULATIONS FALL INTO THE LAST TWO CLASSES OF THE FIRST TABLE IN SECTION III, AND THE REST OF THE PAPER WILL BE DEVOTED TO THESE TYPES OF SIMULATION.

"ALTHOUGH A CONSIDERABLE VOLUME OF WORK HAS BEEN DONE BY OPERATIONS-RESEARCH GROUPS BOTH IN UNITED KINGDOM AND THE UNITED STATES ON THE SUBJECT OF SIMULATION, THERE IS A DEARTH OF PUBLISHED CASE HISTORIES. THE REASON FOR THIS IS PROBABLY THAT A SIMULATION UNDERTAKEN BY AN INDUSTRIAL COMPANY CONTAINS THE KIND OF DATA THAT A COMPANY DOES NOT WISH TO DISCLOSE. FURTHER, THE STRUCTURE OF THE SIMULATION WILL OFTEN GIVE IMPORTANT CLUES TO THE COMPANY FUTURE PLANS.

"THIS IS MUCH TRUER OF A SIMULATION THAN OF MANY OTHER OPERATIONS-RESEARCH STUDIES, BECAUSE, AFTER ALL, A SIMULATION IS A MODEL OF SOME PART OF THE COMPANY'S OPERATIONS AND WILL USUALLY CARRY OVER UNCHANGED MANY OF THE FEATURES OF THE COMPANY'S BUSINESS.

"THE EFFECT OF THIS LACK OF PUBLISHED INFORMATION ON OPERATIONS-RESEARCH SIMULATION IS UNFORTUNATE: THE IMPRESSION IS GIVEN THAT THE USE OF SIMULATION IS RESTRICTED TO SOLVING MATHEMATICAL PROBLEMS, OR ELSE PROBLEMS THAT ARISE IN NUCLEAR PHYSICS OR IN SOME OTHER PURELY TECHNOLOGICAL CONTEXT."

WEST, OF THE RAMO-WOOLRIDGE CORPORATION, HAS THE FOLLOWING TO SAY WITH REGARDS TO THE PRESENT STATE-OF-THE-ART OF SIMULATION AND CONTROL SYSTEMS DESIGN:[23] "IN SUMMARY, COMPUTERS ARE WIDELY USED IN THE ANALYSIS OF CONTROL SYSTEMS. COMPUTER APPLICATIONS TO CONTROL SYSTEM DESIGN ARE PRESENTLY IMPLEMENTATIONS OF THE CUT AND TRY APPROACH. THE FUTURE WILL PROBABLY SEE MORE SOPHISTICATED APPLICATIONS, HOWEVER . . . .

"JUST AS ONE WOULD HOPE TO FIND A BETTER DESIGN PROCEDURE THAN CUT AND TRY, IT IS HOPED THAT A BETTER USE OF THE COMPUTER CAN BE FOUND FOR CONTROL SYSTEM DESIGN. ONE APPROACH IS TO LET THE COMPUTER CONDUCT A SERIES OF EXPERIMENTS AND MODIFY ITS OWN RESPONSE SO AS TO FIND SOME SORT OF AN OPTIMUM. INITIALLY THESE APPROACHES WILL BE LITTLE BETTER THAN THE CUT AND TRY APPROACH OUTLINED ABOVE.

"HOWEVER, AS IMPROVED TECHNIQUES OF SPECIFYING DESIRED RESPONSE AND OF MEASURING DEVIATIONS FROM THE DESIRED RESPONSE ARE EVOLVED, THIS METHOD WILL BECOME MORE USEFUL. DR. BELLMAN'S DYNAMIC PROGRAMMING SEEMS TO OFFER A NEW COMPUTER APPROACH TO CONTROL SYSTEM SYNTHESES."

ALONG SIMILAR LINES ARROW HAS THE FOLLOWING TO SAY:[24] "I WOULD LIKE TO TURN, FOR A FEW MINUTES, TO COMPUTATION. COMPUTING AGAIN, AS A PRACTICAL ART, HAS THE SAME OPEN, TENTATIVE CHARACTER THAT WE HAVE ASSIGNED SPECIFIC

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[23] BY PERMISSION FROM THE ROLE OF COMPUTERS IN ANALYSIS AND DESIGN OF CONTROL SYSTEMS BY G. P. WEST. COPYRIGHTED 1958. INSTITUTE OF RADIO ENGINEERS, TRANSACTIONS ON AUTOMATIC CONTROLS, JULY 1958.

[24] BY PERMISSION FROM DECISION THEORY AND OPERATIONS BY K. J. ARROW. COPYRIGHTED 1957. THE JOURNAL OF THE OPERATIONS RESEARCH SOCIETY OF AMERICA, VOL. 5, NR 6, DECEMBER 1957.

OPERATIONS-RESEARCH PROBLEMS. BEYOND THE SIMPLEST CASES, ANY COMPUTING PROCEDURE IS A PROCEDURE OF SUCCESSIVE APPROXIMATION. WITH COMPUTING MACHINES OF FINITE SIZE, WITH HUMAN BRAINS OF LIMITED CAPACITY, THE MERE STATEMENT THAT WE MUST ACHIEVE AN OPTIMUM IS NOT A SUFFICIENT GUIDE. IN FACT, IN A SENSE, THE PROBLEM OF OPTIMAL DECISION MAKING COULD BE REGARDED AS COMPLETED ONCE WE FORMULATE THE PROBLEM AND ISSUE THE INJUNCTION: 'FIND THE MAXIMUM.'

"FROM A CERTAIN POINT OF VIEW, PROVIDING CERTAIN EXISTENCE AND UNIQUENESS CONDITIONS ARE MET, THIS IS A SOLUTION TO THE ORIGINAL PROBLEM. THE ONLY OBJECTION TO IT IS IN MANY CASES A USELESS SOLUTION, AND MOST OF DECISION THEORY, IN FACT, CONCERNED WITH THE QUESTION OF GETTING SIMPLE, MANAGEABLE EXPRESSIONS FOR THE SOLUTIONS TO VARIOUS SPECIFIC PROBLEMS. NOW, A SOLUTION IN CLOSED FORM IS A NICE THING TO HAVE, ALTHOUGH SOMETIMES IT IS NOT VERY USEFUL. BUT WHAT IS MORE IMPORTANT IS THAT THIS IS NOT THE MOST COMMON SITUATION. FOR THE MOST PART, WE HAVE TO EXPECT THAT OUR SOLUTIONS CAN ONLY BE OBTAINED BY METHODS OF SUCCESSIVE APPROXIMATIONS OF ONE KIND OR ANOTHER. THE CLASSIC METHOD OF FINDING A MAXIMUM IS THE GRADIENT METHOD, OR REALLY, THE GRADIENT FAMILY OF METHODS. THAT IS, IN THE CASE OF AN UNCONSTRAINED MAXIMUM, WE SEEK TO CLIMB UPHILL. NOW, AS I REMARKED EARLIER, THE TYPICAL MAXIMIZATION PROBLEM IN OPERATIONS-RESEARCH IS A PROBLEM OF A CONSTRAINED MAXIMUM. IT IS POSSIBLE IN MANY CASES TO EXTEND THE CLASSICAL GRADIENT METHOD TO THE PROBLEM OF CONSTRAINED MAXIMA BY MAKING USE OF EQUIVALENCE OF CONSTRAINED MAXIMA AND GAMES.

"THE POINT TO BE STRESSED IS THAT WITH THE GRADIENT METHOD OR ANY OTHER METHOD OF SUCCESSIVE APPROXIMATIONS, WE NEVER STRICTLY SPEAKING ACHIEVE THE SOLUTIONS. WE ONLY STOP WHEN WE HAVE DECIDED THAT THE PATH IS STABLE ENOUGH TO WARRANT CESSATION. FURTHER, AS THE PROBLEM BECOMES MORE AND MORE COMPLICATED, THE POSSIBILITY OF ACHIEVING EVEN THIS FIELD OF APPROXIMATION DECREASES. ONE CAN ALWAYS THINK OF REALISTIC PROBLEMS THAT ARE BEYOND THE CAPACITY OF ANY GIVEN SET OF COMPUTING MACHINES. THIS IS NOT TO MINIMIZE THE TREMENDOUS DEVELOPMENTS THAT THE LAST FEW YEARS HAVE SEEN. PROBLEMS THAT COULD HAVE BEEN REGARDED AS WITHIN THE REALM OF PRACTICAL SOLUTION NOW FALL INTO THIS REALM. ALL THIS SEEMS TO DO, OF COURSE, IS TO WHET OUR APPETITE FOR STILL BIGGER PROBLEMS.

"THERE ARE SEVERAL APPROACHES TO THE COMPUTATIONAL PROBLEM. ONE IS TO EXAMINE CLOSELY THE SPECIAL FEATURES OF THE PARTICULAR PROBLEM, TRY TO DEVISE COMPUTATIONAL METHODS THAT WILL EXPLOIT THOSE SPECIAL FEATURES AND SO BE EFFICIENT. (FROM A THEORETICAL VIEWPOINT, THE DISCOVERY OF NEW COMPUTATIONAL METHODS IS ALSO A DECISION-THEORETICAL PROBLEM, AND YET IT IS NOT ONE THAT HAS BEEN MUCH HANDLED OR WELL HANDLED. IT REALLY BECOMES A CREATIVE ACT, WITHIN THE PRESENT REALM OF OUR EXPERIENCES.) THE SIMPLEX AND OTHER METHODS IN LINEAR PROGRAMMING AND THE METHOD OF FUNCTIONAL EQUATIONS IN DYNAMIC PROGRAMMING ARE EXAMPLES OF THIS PROCEDURE. AS WE GET TO MORE AND MORE SPECIFIC PROBLEMS WE FIND CERTAINLY MORE AND MORE TOE-HOLDS FOR COMPUTATIONAL METHODS. BUT USUALLY THERE ARE SEVERE LIMITATIONS TO THEIR APPLICABILITY.

"A Second technique has been to treat, not the original problem, but a simpler problem that can be managed. It is hoped that enough of the features of the original problem will remain so that the solution us useful. This method is, in fact, universal. I doubt if there is any real problem that anybody has handled which does not involve a suppression of at least some complications. Again, the question of simplification is a creative act.

"Still a third technique is to abandon the search for the optimum and seek instead to find satisfactory procedures. That is, we guess at a procedure, and then seek to determine, by real or simulated experience, what its operating characteristics are. If they strike us as being good enough, we use them. Otherwise, we search for more and better methods. In a sense, this last procedure, which ties in closely with what psychologists call the achievement of a level of aspiration, is a dynamic counterpart to the seeking of an optimal method. In fact, it is a method of successive approximations applied to the choice of a computing method. Once a satisfactory method has been found, there is always a tendency to try to improve it. Presumably this process will continue until it converges to the optimum. Like all methods of successive approximations, the convergence will take infinitely long, if indeed it converges at all.

"To sum up: At any stage in our operations research, we deal with problems more limited than those we know are real - more limited in time if not in other ways. Our problem must be stated as if it were closed, so that it can be solved, and yet its elements must contain within themselves the possibility of fitting into a larger model. The objective function must reflect the fact that the problem will have implications for the future. Among other things, we must assign valuation to information-gathering and recording activities in our short run or proximate objective function. We must assign values to the physical outcome of the current decision situation, which somehow reflect our estimates, however, formed of their implications for the future. We must, in short, build in the possibility of learning into our decision models. Further, even then, we frequently cannot solve our decision problems in the sense of genuinely achieving an optimum with a small expenditure of computing effort. So we are approximating in many different directions. This kind of view of the world may seem somewhat disturbing to some. One likes the idea that there is a finite solution to problems - that they all can be solved with expenditures of sufficient mental energy. But the whole history of mathematics and science is a disproof of any such simple notion. And personally, I now find it rather attractive that the world is an open one, and that there always will be new problems to solve. At no stage will we be able to rest content. As Goethe says: 'Who ever strives, him can we save.'"

In conclusion, a thought from Taylor[25] seems appropriate: "Finally, let me brush on what I feel are two general problems or obstacles in non-linear control. One is the lack of useful existing mathematics or mathematical tools. Here I feel that there will be required a bootstrap type operation between mathematicians and operating control engineers. The common meeting ground may well be computer experimentation."

Kron's approach to the system complexity problem is presented in his article "Tearing, Tensors, and Topological Models:[26] "Today's engineers, operating in an age of accelerated technology, are being confronted with increasingly more complex and interconnected physical structures (systems), which require analysis and solution. The electronic computer was designed to fill this need. However, many problems are constantly several steps ahead of the computers, overshooting the latter's capacity to produce economical solutions.

"In this unequal race between the size of problems and the capacity of computers, a succession of blind alleys is encountered. First of all, many important classes of problems cannot even be stated mathematically. Function theory can take care of only the simplest geometrical shapes, such as spheres and cylinders. Where it is possible to formulate the problem, the physical labor of writing down the hundreds, or perhaps thousands, of equations is often simply prohibitive. Moreover, even it is possible to write the problem down, it is often found that the available computers are not large enough, or fast enough, to carry through the computations economically.

"The chasm between the immensity of problems and the limited capacity of the computers can, however, be bridged somewhat - for certain classes of problems - by approaching the situation from a new angle. Instead of asking for a still larger computer, the proposed method of attack tears apart the otherwise unmanageable physical or economic system into several smaller parts, solves each part separately, then interconnects the partial solutions into those of the original system. Tearing a problem apart also eliminates the necessity of stating or defining the problem mathematically in its entirety. A formulation of each segment only, and a statement of their interconnection suffices. Thus the method of tearing also opens up the possibility of solving a class of methematically-undefinable engineering problems.

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[25] By permission from Problems in Non-Linearity in Adaptive Self-Optimizing Systems by C. F. Taylor. Copyrighted 1958. Institute of Radio Engineers Transactions on Automatic Control, July 1958.

[26] By permission from Tearing Tensors and Topological Models by G. Kron. Copyrighted 1957. The American Scientist, Vol. 45, Nr 5, Dec. 1957.

"PHYSICAL AND FUNCTIONAL TEARING

IT SHOULD NOT BE BELIEVED, HOWEVER, THAT AS SOON AS THE QUESTION OF PIECEWISE SOLUTION OF PROBLEMS IS RAISED, ONE CAN PROCEED AND IMMEDIATELY ACCOMPLISH THAT TASK. THE SYSTEMATIC INTERCONNECTION OF PARTIAL SOLUTIONS IS A PROCEDURE THAT MANY ENGINEERS HAVE THOUGHT TO BE IMPOSSIBLE, OR AT LEAST EXTREMELY DIFFICULT AND LABORIOUS, HENCE A THANKLESS AND FRUITLESS UNDERTAKING. IN TRIVIAL CASES, AS IN MECHANICAL STRUCTURES COUPLED AT ONE OR A FEW POINTS, EACH PROBLEM IS SOLVED ON ITS OWN MERIT, WITHOUT ANY ATTEMPT AT SYSTEMATIZATION OR GENERALIZATION. SCIENTIFIC LITERATURE ALSO CONTAINS SCATTERED REFERENCES TO CUT-AND-TRY PROCEDURES FOR SOMEHOW PIECING TOGETHER SPECIAL TYPES OF PARTIAL SOLUTIONS. THE METHOD OF TEARING, TO BE PRESENTED HERE, MEETS THAT EVER PRESENT PROBLEM HEAD-ON.

"DURING HIS PROFESSIONAL CAREER THE AUTHOR HAS BEEN ENGAGED IN DEVELOPING A NEW ENGINEERING TOOL, A NEW METHOD OF LARGE-SCALE THINKING, WHICH WOULD ENABLE THE ENGINEER TO ORGANIZE, ANALYZE, AND - IF POSSIBLE - SOLVE HIGHLY COMPLEX STRUCTURE AND SYSTEMS IN EACH STAGE , AND IN A ROUTINE MANNER. TO MEET THAT PURPOSE, HE HAS GRADUALLY DEVELOPED A THREE-PRONGED TOOL, WHICH UTILIZES THE THEORY OF TENSORS, TOPOLOGICAL MODELS IN THE FORM OF ELECTRICAL CIRCUITS, AND THE TEARING APART OF SUCH CIRCUITS PHYSICALLY OR FUNCTIONALLY, (OR BOTH WAYS SIMULTANEOUSLY) INTO SMALLER PIECES. (THE ELECTRIC CIRCUITS ARE MERELY DRAWN ON A SHEET OF PAPER AND NEED NOT BE PHYSICALLY REALIZABLE). THE PHYSICAL SUBDIVISION LEADS TO EASIER AND FASTER DIGITAL OR ANALOGUE MANIPULATIONS, WHILE THE FUNCTIONAL SUBDIVISION AIMS AT MORE CORRECT AND MORE VISUALIZABLE PHYSICAL CONCEPTS ESPECIALLY IN MULTI-ENERGY SYSTEMS.

"FOR TWO DECADES THE AUTHOR HAD TO BE CONTENT WITH CONCENTRATING UPON THE ORGANIZATION AND SETTING-UP OF EQUATIONS OF PERFORMANCE FOR A LARGE VARIETY OF SYSTEMS IN A PIECEWISE MANNER. ONLY LATELY HAS THE TOOL ADVANCED TO SUCH A STAGE THAT THE MISSING LINKS BETWEEN SETTING-UP EQUATIONS PIECEWISE, AND SOLVING EQUATIONS PIECEWISE, HAVE BEEN DISCOVERED AND PUT TO PRACTICAL USE.

"LINEAR AND NON-LINEAR PHYSICAL PROBLEMS NUMERICALLY AS WELL AS ANALYTICALLY (BY FACTORING OUT PARAMETERS), TIME-VARYING AND EIGENVALUE PROBLEMS, ALSO OPTIMIZATION OF PHYSICAL AND ECONOMIC SYSTEMS, ALL IN SMALL INSTALLMENTS, MULTI-ENERGY STABILITY PROBLEMS, TORN FRACTIONALLY AS WELL AS PHYSICALLY, HAVE BEEN SOLVED. THE INTERCONNECTED SOLUTIONS ARE IN ALL CASES AS EXACT, (OR AS APPROXIMATE), AS IF THE SYSTEM HAD NOT BEEN TORN APART, BUT SOLVED IN ONE PIECE.

"LIMITATIONS OF PIECEWISE SOLUTIONS

DURING THE LAST FEW YEARS THE AUTHOR HAS ALSO CLARIFIED SOME OF THE MATHEMATICAL AND PHYSICAL ASPECTS OF THE METHOD OF TEARING. TWO DEFINITE LIMITATIONS CAN ALREADY BE DISCERNED. FIRST OF ALL, A GIVEN SET OF EQUATIONS CANNOT BE TORN. (PARTIONING A MATRIX IS A CONCEPT DIFFERENT FROM TEARING.) THE EQUATIONS MUST BE ACCOMPANIED BY THE ORIGINAL SYSTEM (OR RATHER BY A GRAPHICAL MODEL OF IT) AND IT IS THE LATTER PART THAT IS TORN APART, NOT THE

EQUATIONS. THE REASON IS THAT THE GRAPH CONTAINS MORE INFORMATION ABOUT THE ORIGINAL SYSTEM THAN THE EQUATIONS.

"SECONDLY, IT IS IMPORTANT THAT THE NUMBER OF LINKS TORN SHOULD BE MUCH SMALLER BY COMPARISON THAN THE NUMBER OF VARIABLES OF THE ENTIRE SYSTEM, (AS THE CONSTRAINED FORCES APPEARING AT THE CUT ALSO MUST BE SOLVED FOR). HENCE, IF THE SYSTEM IS TIGHTLY COUPLED, THE METHOD OF TEARING SHOULD NOT BE USED FOR THE SOLUTION. (FOR INSTANCE, THE N-BODY PROBLEM OF GRAVITATION, OR A NETWORK IN WHICH EVERY POINT IS CONNECTED TO EVERY OTHER POINT) . . . .

"THEN ON SOME FUTURE OCCASION, ONE CAN INTERCONNECT SEVERAL OF THE SOLVED SYSTEMS ON FILE, TO FORM SOLUTIONS OF STILL LARGER SUPERSYSTEMS. SEVERAL OF THE SOLVED SUPERSYSTEMS IN TURN CAN BE INTERCONNECTED IN THE STILL MORE DISTANT FUTURE. THE SOLUTIONS GROW LIKE A PYRAMID, LAYER UPON LAYER, BE IT AN EIGENVALUE SOLUTION, OR AN OPTIMIZATION, OR STRAIGHT NUMERICAL INVERSION. NEVERTHELESS, THE OVER-ALL SOLUTION REMAINS EXPRESSED AT ALL TIMES IN THE FORM OF SMALL, MANAGEABLE MATRICES.

"IF THE ENGINEER DECIDES TO ALTER A SMALL PORTION OF HIS SYSTEM, ALL SOLUTION-MATRICES ARE RETAINED EXCEPT THE ONE WHICH IS TO BE CHANGED. THERE IS NO NEED TO START THE PROCESS OF SOLUTION OF THE ENTIRE SYSTEM ALL OVER AGAIN FOR EACH SMALL CHANGE, AS IS USUALLY THE CASE WITH ONE-PIECE SOLUTIONS.

"SINCE EVERY STEP IN THE SOLUTION PROCESS REPRESENTS SOME PHYSICALLY EXISTING EXCITATION (OR RESPONSE) OR PHYSICALLY EXISTING STRUCTURES, THE CORRECTNESS OF THE PARTIAL RESULTS CAN CONSTANTLY BE CHECKED BY APPLYING PHYSICAL TESTS WITH KNOWN RESULTS. ALSO, IN A PIECEMEAL INVERSION, THE ROUND-OFF ERRORS DO NOT ACCUMULATE AS FAST AS IN A ONE-PIECE GAUSSIAN INVERSION . . . .

"AS MANY OF THE IMPORTANT PARTIAL DIFFERENTIAL EQUATION OF QUANTUM PHYSICS ARE LINEAR (SCHRODINGER, DIRAC, KLEIN-GORDON, ETC., EQUATIONS) TOPOLOGICAL MODELS OPEN UP A NEW METHOD OF ATTACK FOR THEIR SOLUTION. THE THEORY OF LINEAR VECTOR SPACES OFFERS LABOR-SAVING DEVICES TO REDUCE THE LARGE NUMBER OF EIGENVALUES OCCURRING IN FIELD PROBLEMS TO A PRACTICABLE SMALL NUMBER, WITHOUT REDUCING THE NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM . . . .

"TIME-VARYING SOLUTIONS

WHEN LONG TIME-INTERVALS ΔT ARE ASSUMED, THE IMPLICIT RECURRENCE FORMULAE OF VON NEUMANN REQUIRE THE INVERSION OF A MATRIX AT EACH STEP BY SOME ITERATIVE PROCEDURE. THE ALMOST-DIAGONAL FORM OF A FACTORIZED INVERSE MATRIX - ESTABLISHED ONCE AND FOR ALL BY THE METHOD OF TEARING - REQUIRES ONLY A SIMPLE SUBSTITUTION AT EACH STEP, WITHOUT THE DANGER OF INSTABILITY DUE TO ITERATION.

"ANOTHER APPROACH IS ALSO POSSIBLE. WHEN THE TIME AND SPACE VARIABLES CAN BE SEPARATED, THE SPATIAL FIELD NETWORKS CAN BE SOLVED (BY THE METHOD OF TEARING) AS EIGNEVALUE PROBLEMS, WITH A PRACTICAL NUMBER OF THE LOWER OR HIGHER EIGNEVALUES (SEPARATION-CONSTANTS) AND EIGENVECTORS. THE TIME VARIATION OF THE WAVES ARE THEN EXPRESSED BY LAPLACE TRANSFORMS. BOTH SCALAR AND

VECTOR FUNCTIONS CAN BE SO EVALUATED.

"THE NUMEROUS TYPES OF AUTOMATIC-CONTROL SYSTEMS, AS WELL AS PROBLEMS IN THE SYNTHESIS OF NETWORKS - WITH THEIR REQUIREMENTS TO CALCULATE A LARGE NUMBER OF ZEROS AND POLES - SEEM TO OFFER A FERTILE GROUND FOR APPLICATIONS OF THE METHOD OF TEARING . . . .

"SUCH SUCCESSFUL PIECEWISE SOLUTIONS BY PHYSICAL AND FUNCTIONAL, FINITE AND INFINITESIMAL TEARING, POINT THE WAY TO THE POSSIBILITY OF AN EVENTUAL PIECEWISE ATTACK UPON HIGHLY COMPLEX SYSTEM-STABILITY PROBLEMS WHICH INVOLVE MORE THAN TWO TYPES OF INTERACTING ENERGIES, SUCH AS, FOR INSTANCE, THE STABILITY OF MAGNETO-HYDRODYNAMIC SYSTEMS. OF COURSE THERE ARE MANY OBSTACLES ALONG THE WAY WHICH ARE YET TO BE OVERCOME, BEFORE SUCH INTRICATE SYSTEM PROBLEMS CAN BE TORN BOTH PHYSICALLY AND FUNCTIONALLY . . . .

TEARING ECONOMIC AND OTHER PROBLEMS

IN ENGINEERING AND PHYSICS THE GIVEN SYSTEM ITSELF DEFINES AN UNDERLYING TOPOLOGICAL STRUCTURE, THAT IS, A CIRCUIT-MODEL. HENCE, THE ANALYSIS AND SOLUTION OF PHYSICAL SYSTEMS BY TEARING IS A NATURAL, SELF-EVIDENT ATTRIBUTE. HOWEVER, THERE ARE MANY MATHEMATICALLY-DEFINABLE PROBLEMS WHERE THE EXISTENCE OF AN UNDERLYING TOPOLOGICAL STRUCTURE IS NOT OBVIOUS, BUT NEVERTHELESS CAN BE CONJECTURED. IF A SEARCH IS MADE FOR A STRUCTURE IT USUALLY CAN BE FIRMLY ESTABLISHED.

SUCH STRUCTURES EXIST, FOR INSTANCE, IN MANY OPERATIONS RESEARCH AND IN A LARGE CLASS OF OTHER ECONOMIC PROBLEMS. WHEREVER MANUFACTURED GOODS OR OTHER ENTITIES ARE ROUTED ALONG PREASSIGNED PATHS, THE UNDERLYING TOPOLOGICAL STRUCTURE OF THE PROBLEM IS RATHER WELL DISCERNED, EVEN THOUGH IT MIGHT BE OBSCURED BY THE SUPERIMPOSED ALGEBRAIC STRUCTURE. IN GENERAL, PROBLEMS FORMULATED IN TERMS OF BLOCK-DIAGRAMS OR FLOW-DIAGRAMS DO POSSESS AN ALREADY-DEFINABLE TOPOLOGICAL STRUCTURE. THE AUTHOR HAS SUCCEEDED IN OPTIMIZING 'LINEAR PROGRAMMING' OF TRANSPORTATION PROBLEMS PIECEWISE.

"AT THE OPEN PANEL DISCUSSION OF THE 1957 PGAC SYMPOSIUM ON NON-LINEAR CONTROL, A SPEAKER FROM THE FLOOR REPEATED AN ANCEDOTE HE HAD HEARD ' . . . THAT A VERY DIFFICULT MISSILE PROBLEM IS TO TAKE AN OBLATE SPHEROID, TRANSMITTED FROM A MOVING TARGET THROUGH A MOVING ATMOSPHERE AND HIT IT, OR HAVING TRANSMITTED IT FROM A MOVING PLATFORM, HURL IT THROUGH A MOVING TARGET . . .' THE COMPUTER THAT YOU BUILD TO DO THIS WOULD CERTAINLY FILL THIS ROOM IF YOU DID A GOOD JOB. (YET) YOU CAN TAKE A HALF-WITTED HALFBACK AND DO IT NINE TIMES OUT OF TEN ANY SATURDAY AFTERNOON ALL FALL. (THE POINT IS) THAT LAST YEAR, SOME PLACE, THIS HAPPENED TO MANY OF THESE OPERATORS OF MUCH BETTER COMPUTERS THAN YOU HAVE IN YOUR ENGINEERING DEPARTMENTS."

WEAPON SYSTEM

A weapon system may be considered in terms of its constituent parts in accordance with Goode and Machol[27] as: Inputs, communication, logical control, reflexive control, handling and outputs. Reflexive controls refer to error type servos etc., whereas logical controls imply the capability of decision making. The steps in system design as conceived by Goode and Machol are shown in Fig. 1-2.

The synthesis and analysis of a weapon system utilizing modern high speed computing machines may be considered in the following four main phases.

1. Over-all system study of the complex battle (multiple targets and target interceptors) against present and future targets to determine the design criteria for target interceptor weapon system.

2. Synthesis and analysis of design equations of proposed weapon system to satisfy requirements of 1. (Pre-hardware.) This area includes purely mathematical simulation.

3. Analysis of physically realized system. Analysis based on laboratory tests of hardware as it becomes available. Phase 3 will serve to make phase 2 analysis studies continuously more realistic.

4. Over-all system analysis of physically realized weapon system (based on phase 3 results). This phase entails replacing the theoretical weapon system capability of phase 1 by the actually engineered system.

The major distinction between phase 2 and 3 and phase 3 and 4 is that a single unit of the over-all system is considered, whereas phases 1 and 4 consider multiple units and utilize the concepts of game theory.

For the purpose of the mathematical simulation of a physical system, the system is broken down into major and minor subsystems according to distinct functional relations. The ideas of these four phases are shown in Figure 1-3.

FLOW DIAGRAMS

In order to expedite the communication of system concepts, the flow diagram has evolved. The verbal flow diagram gives a general picture of the

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[27] By permission from Control Systems Engineering by H. H. Goode and and R. E. Machol. Copyrighted 1957. McGraw-Hill Book Company, Inc.

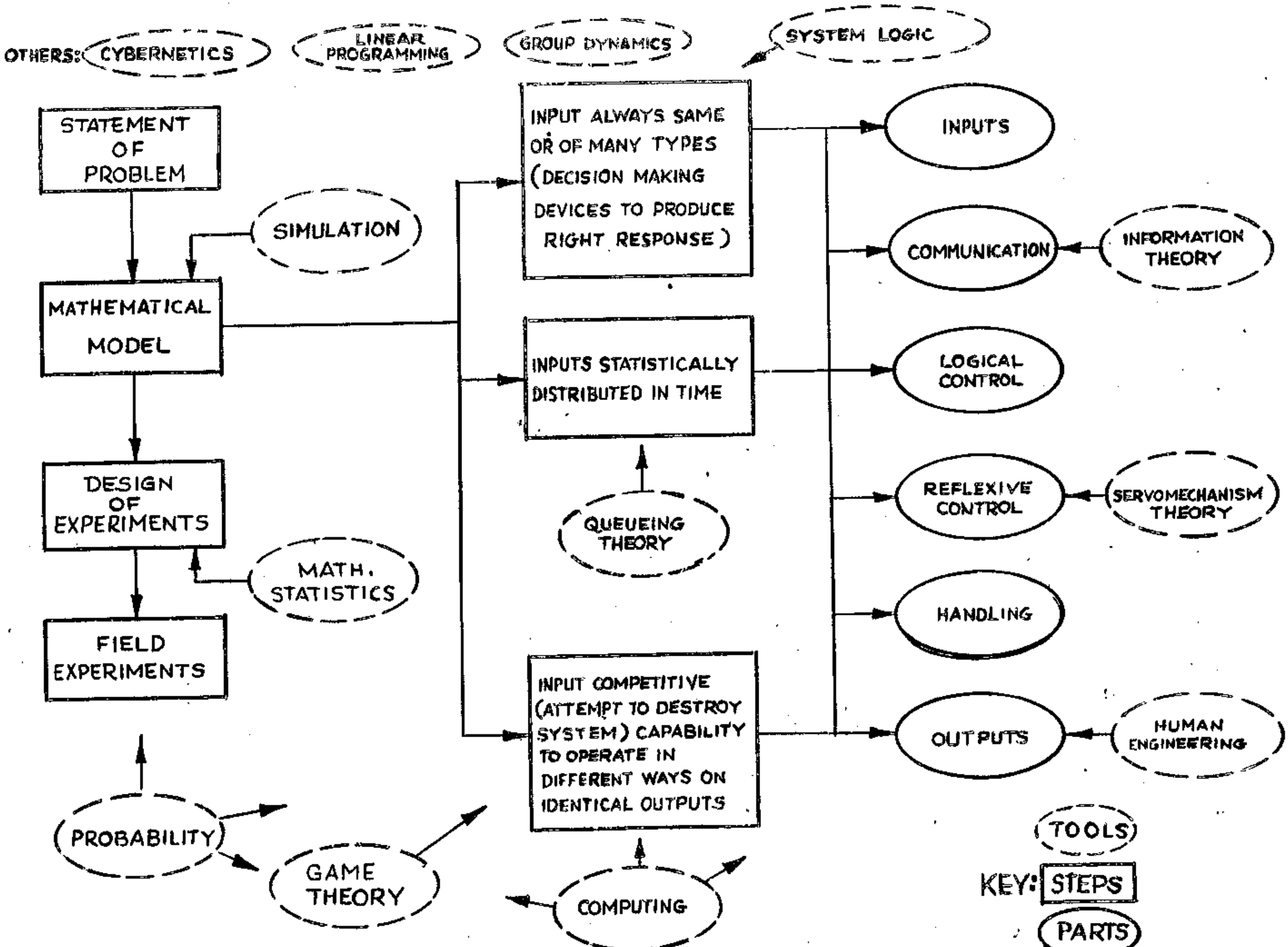


Fig. 1-2 STEPS IN SYSTEM DESIGN AS CONCEIVED BY GOODE AND MACHOL

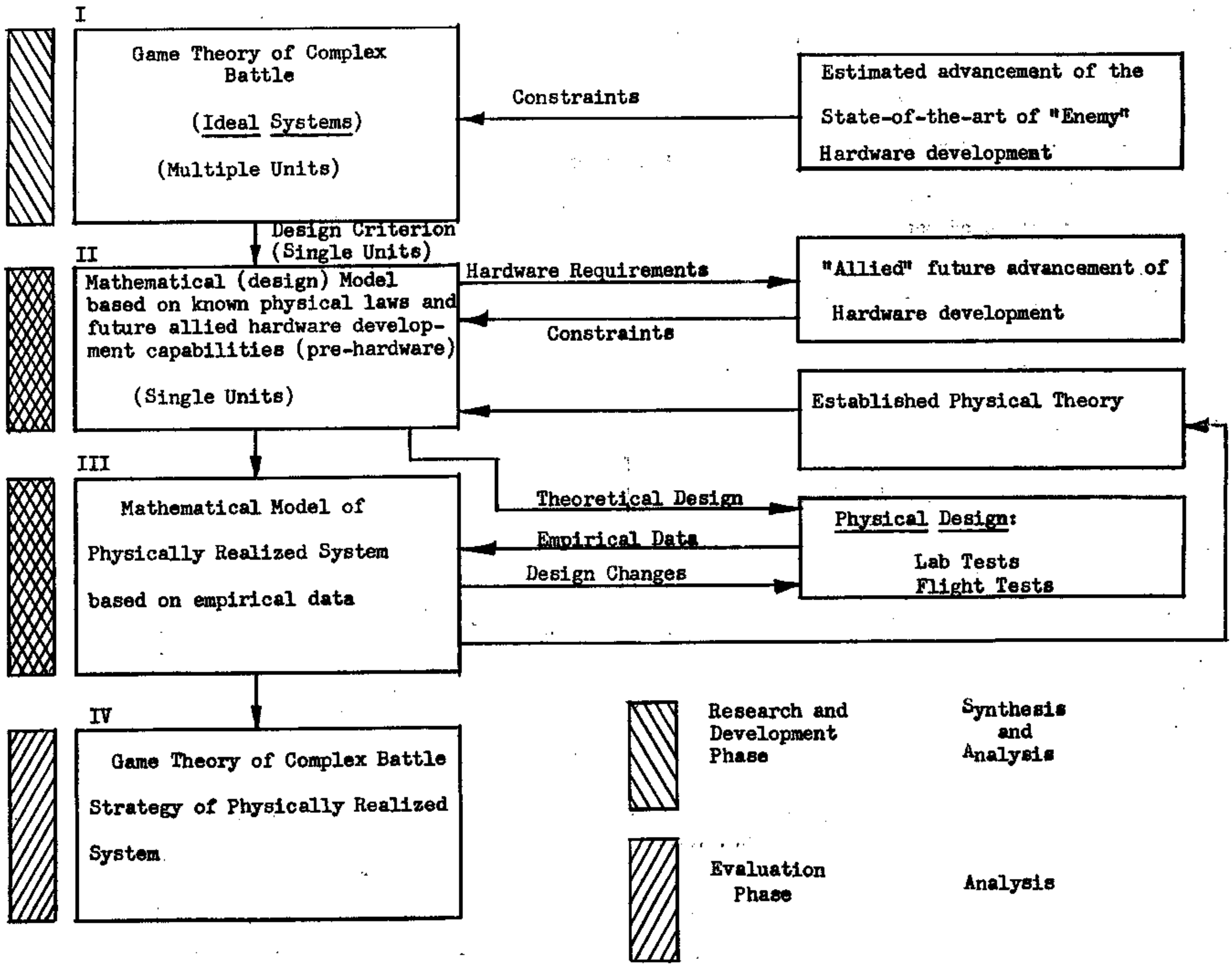


Fig. [illegible] Simulation Steps In Weapon Systems

SYSTEM BLOCKS AND THE FLOW OF VARIABLES BETWEEN BLOCKS IN TERMS OF VERBAL PHYSICAL STATEMENTS. THE MATHEMATICAL MODEL FLOW DIAGRAM SEPARATES THE MANY MATHEMATICAL EQUATIONS INTO SUBSETS OF EQUATIONS SO THAT THE FUNCTIONAL RELATION BETWEEN THE VARIOUS SUBSETS OF EQUATIONS IS EVIDENT (FOR SYSTEMS NOT TOO INTRICATELY COUPLED). THE BLOCK DIAGRAM APPROACH SHOWS THE FORCING FUNCTIONS OF THE VARIOUS BLOCKS OF EQUATIONS. FOR EXAMPLE, A SIX DEGREE OF FREEDOM GUIDED MISSILE SIMULATION OVER A ROTATING SPHEROIDAL EARTH IS BROKEN DOWN INTO A SMALL NUMBER OF BASIC MATHEMATICAL OPERATIONS SUCH AS SUMMATION, INTEGRATION, MULTIPLICATION, OR INTO GROUPS OF OPERATIONS SUCH AS COORDINATE TRANSFORMATION BLOCKS, DIRECTION COSINE MATRIX GENERATION BLOCKS, ETC., AS SHOWN IN FIGURES 1-4-A AND 1-4-B. THE OTHER IMPORTANT BLOCKS OF THE SIMULATED MISSILE SYSTEM ARE THE GENERATION OF THE THREE FORCES; AERODYNAMIC, GRAVITATIONAL, AND THRUST; SOLUTION OF THE AIR FRAME DYNAMICS (TRANSLATION AND ROTATION); PLUS THE SIMULATION OF THE GUIDANCE AND CONTROL SUBSYSTEMS.

THIS TYPE OF SIMULATION FOR RIGID MODELS IS "OLD STUFF" TO MANY LABORATORIES AND THE MAIN PROBLEMS HERE ARE MORE COMPUTERS WITH HIGHER SPEED AND ACCURACY. HOWEVER, THE DEVELOPMENT OF THE PROBABILISTIC MODELS FOR THE AIR-FRAME, GUIDANCE AND CONTROL COMPLEX, IS THE NEXT LARGE ADVANCEMENT TO BE ACHIEVED. THE FIRST APPROACH IS AN EXTENSION OF THE PRESENT SIMULATION TECHNIQUES OF SIMULATION OF RADAR NOISE, TO A MORE REALISTIC SIMULATION INCLUDING MANY MORE OF THE SYSTEM STOCHASTIC VARIABLES AND PARAMETERS AS SHOWN IN FIGURE 1-5. THIS APPROACH REQUIRES MORE COMPUTING CAPACITY THAN IS CURRENTLY AVAILABLE IN THE TYPICAL WEAPON SYSTEM SIMULATION LABORATORY TODAY. FROM A STATISTICAL STUDY OF THE COMBINED EFFECT OF THE MANY STOCHASTIC VARIABLES A FUNCTIONAL APPROACH MAY BE FEASIBLE.

FUNCTIONAL FLOW DIAGRAMS (OR MODELS) OF INTEREST TO SYSTEMS STUDIES ARE:

1. PHYSICAL SYSTEM.

2. SIMULATED SYSTEM.

THE PHYSICAL AND SIMULATED SYSTEMS MAY BE FURTHER DEPICTED BY:

1. VERBAL FLOW MODELS.

2. MATHEMATICAL MODELS.

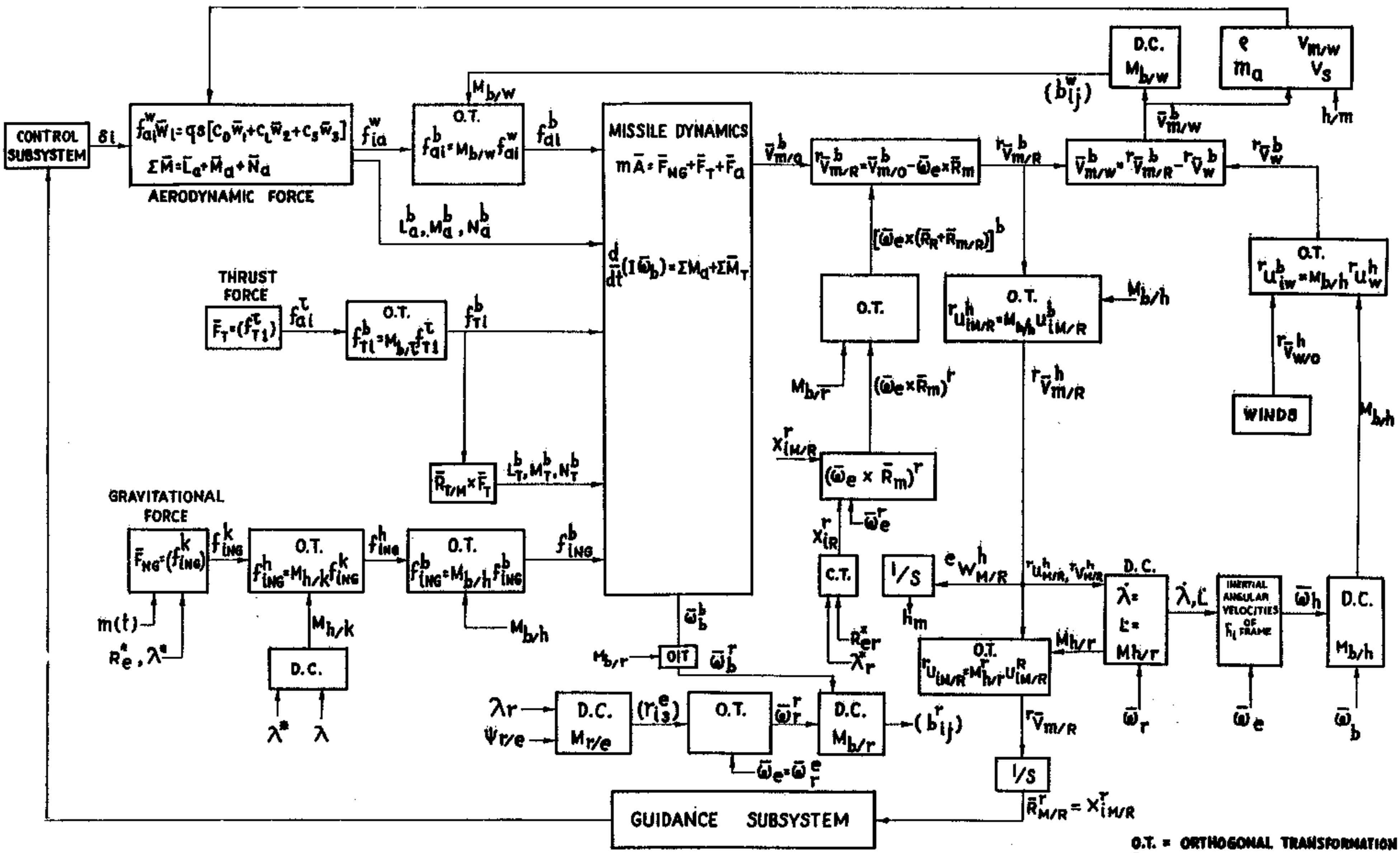
PHYSICAL SYSTEM

AS STATED ABOVE, TWO FLOW CHARTS ARE OF VALUE RELATING TO THE PHYSICAL SYSTEM, THESE ARE:

1. VERBAL PHYSICAL SYSTEM MODEL.

2. MATHEMATICAL MODEL OF PHYSICAL SYSTEM.

THE MATHEMATICS OF THE PHYSICAL SYSTEM SHOULD DEFINE THE SYSTEM DESIGN EQUATIONS AND ALSO INCLUDE THE CHARACTERISTICS OF THE PHYSICAL COMPONENTS

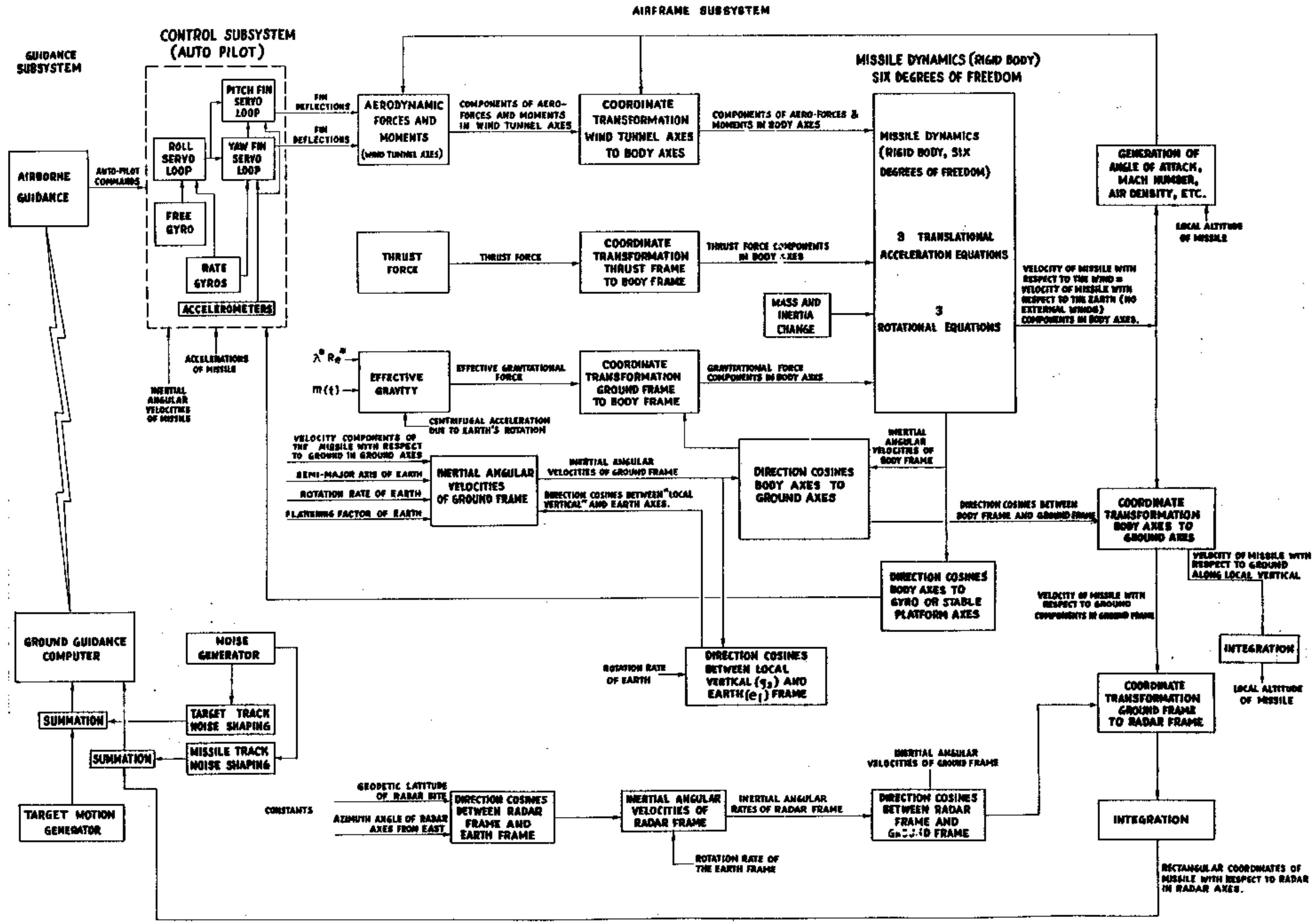


O.T. = ORTHOGONAL TRANSFORMATION
D.C. = DIRECTION COSINE MATRIX

DETERMINISTIC SYSTEM BLOCK DIAGRAM

FIGURE 1-4 a

FIGURE I-4 b

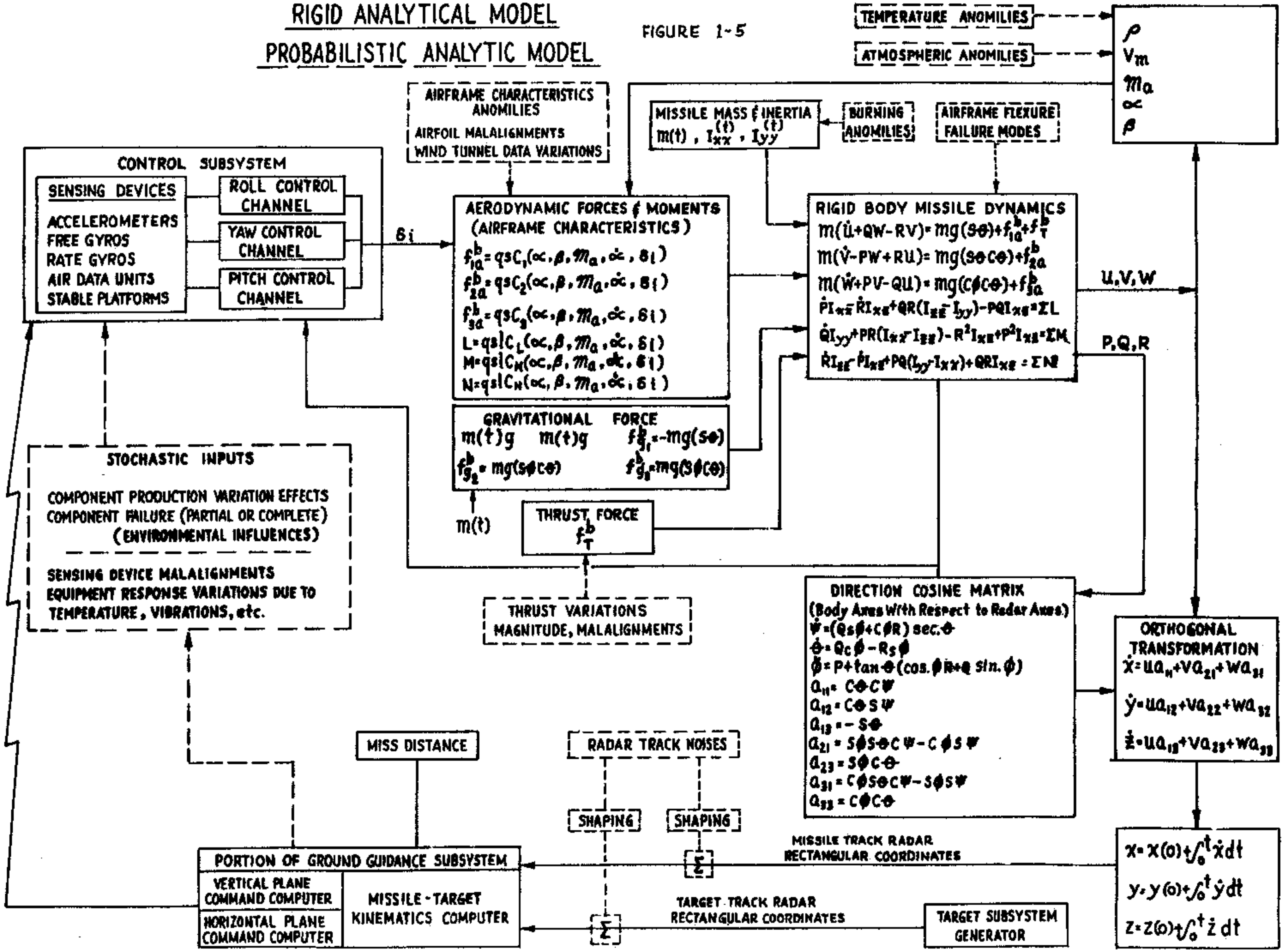


VERBAL BLOCK DIAGRAM OF A SIMULATED MISSILE SYSTEM

RIGID ANALYTICAL MODEL

PROBABILISTIC ANALYTIC MODEL

FIGURE 1-5



USED TO INSTRUMENT THE EQUATIONS (I.E. THE TRANSFER CHARACTERISTICS AND THE NOISE).

SIMULATED SYSTEMS

TWO FLOW CHARTS ARE OF VALUE RELATING TO THE SIMULATED SYSTEM, THESE ARE:

1. VERBAL SIMULATED SYSTEM FLOW CHART.

2. MATHEMATICAL MODEL OF THE SIMULATED SYSTEM.

IT MAY BE OF VALUE TO ALSO HAVE OVERLAYS OR A FURTHER BREAKDOWN OF THE SYSTEM INTO GUIDANCE MODES, SUCH AS:

1. LAUNCH GUIDANCE.

2. MIDCOURSE GUIDANCE.

3. TERMINAL GUIDANCE.

THUS THE PHYSICAL SYSTEM AND SIMULATED SYSTEM BLOCK DIAGRAMS ABOVE WOULD BE ALSO AVAILABLE IN REDUCED FORM SHOWING ONLY THE EQUIPMENT OR EQUATIONS PREVAILING DURING THAT PARTICULAR GUIDANCE MODE.

THE DIFFERENCE BETWEEN THE PHYSICAL SYSTEM AND THE SIMULATED SYSTEM IS OBVIOUS AFTER A MOMENTS REFLECTION, FOR EXAMPLE, FREE - GYRO GIMBAL PICK-OFF ANGLES ARE AVAILABLE DUE TO THE PHYSICAL MOTION OF THE MISSILE IN ACTUAL FLIGHT, WHEREAS IN THE SIMULATED SYSTEM, THE ANGULAR MOTION OF THE AIRFRAME MUST BE COMPUTED FROM THE ROTATIONAL EQUATIONS OF MOTION BASED ON WIND TUNNEL DATA. THUS, KNOWING THE ROTATION OF THE AIRFRAME AND THE GYRO ORIENTATION, TWO OF THE EULER ANGLES ORIENTING THE TWO FRAMES MAY BE COMPUTED, AND HENCE THE IDEAL GIMBAL ANGLE PICK-OFFS ARE SIMULATED.

VERBAL FLOW MODELS

THE VERBAL FLOW CHARTS ARE OF VALUE TO GIVE A DESCRIPTION OF THE OVERALL SYSTEM FLOW OF THE DESIGN VARIABLES. THIS BIRDS-EYE GLIMPSE REVEALS GENERAL DESIGN CONCEPTS WITHOUT REQUIRING A KNOWLEDGE OF THE MATHEMATICAL NOMNECLATURE.

FOR THE PURPOSE OF THE MATHEMATICAL SIMULATION OF A PHYSICAL SYSTEM, THE SYSTEM IS BROKEN DOWN INTO MAJOR AND MINOR SUBSYSTEMS ACCORDING TO FUNCTIONAL RELATIONS. THE WEAPON SYSTEM IS CONSIDERED TO CONSIST OF THREE MAJOR SUBSYSTEMS AS SHOWN IN FIG. 1-6.

A. TARGET MAJOR SUBSYSTEM.

B. TARGET INTERCEPTOR MAJOR SYBSYSTEM.

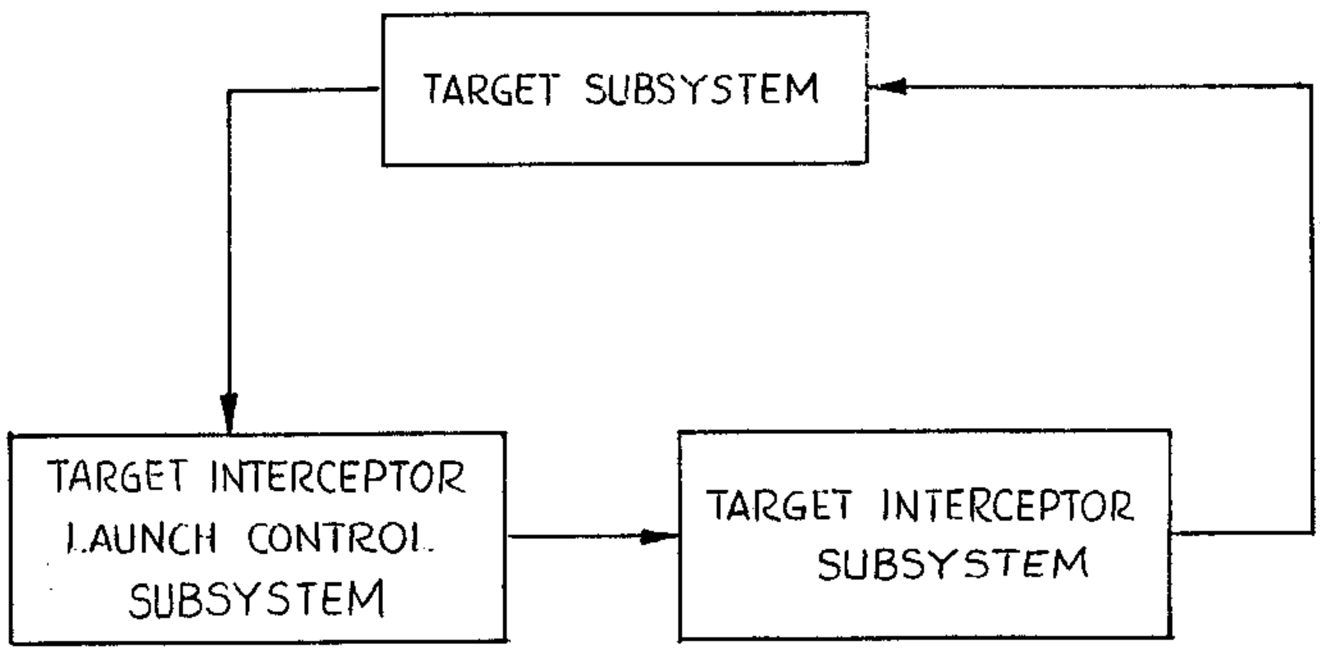


FIGURE 1-6 WEAPON SYSTEM MAJOR SUBSYSTEMS

c. Target interceptor launch control major subsystem.

The three major subsystem breakdown enumerated above will be given the following broad and general interpretation for the purposes of this report (the writer being fully aware that there are almost as many different concepts of how a weapon system should be broken down as there are missilemen).

Target-Interceptor Launch Control Subsystem

The primary function of this subsystem is to determine the time the target interceptor shall be released from its launcher to carry out its mission of delivering itself sufficiently close to a designated target. The launch control function may be broken down into three (more or less) sub-functions to be loosely designated as:

1. Target detection and filtering of non-threats.

2. Target trajectory fitting and prediction.

3. Interceptor selection.

The target detection may be based on many years political build-up to determine such and such a country is an enemy or on detecting incoming enemy missiles in an anti-missile defense system. The trajectory fitting may involve geodetic surveys of earth fixed targets or radar tracking of invading vehicles. In the former case, prediction is an easy matter, as earth's rotation rate is known. The final phase, interceptor selection, may involve moving the target-interceptor by rail, ship, etc., to advanced lines, the final aiming and firing at earth fixed or moving targets. In practice, the logistics - mathematics (in its infancy) is not included in the simulation study of target and target-interceptor motion. However, for air defense studies, the logic, decision making etc., which are part of the physical system may also be simulated.

Target Interceptor Major Subsystem

This subsystem is the guided missile and its warhead. The target-interceptor major subsystem may be considered as consisting of three subsystems:

1. Vehicle and Environment subsystem.

2. Control subsystem.

3. Guidance subsystem.

as shown in Fig 1-7 and Fig. 1-8 for the physical model and the simulated model.

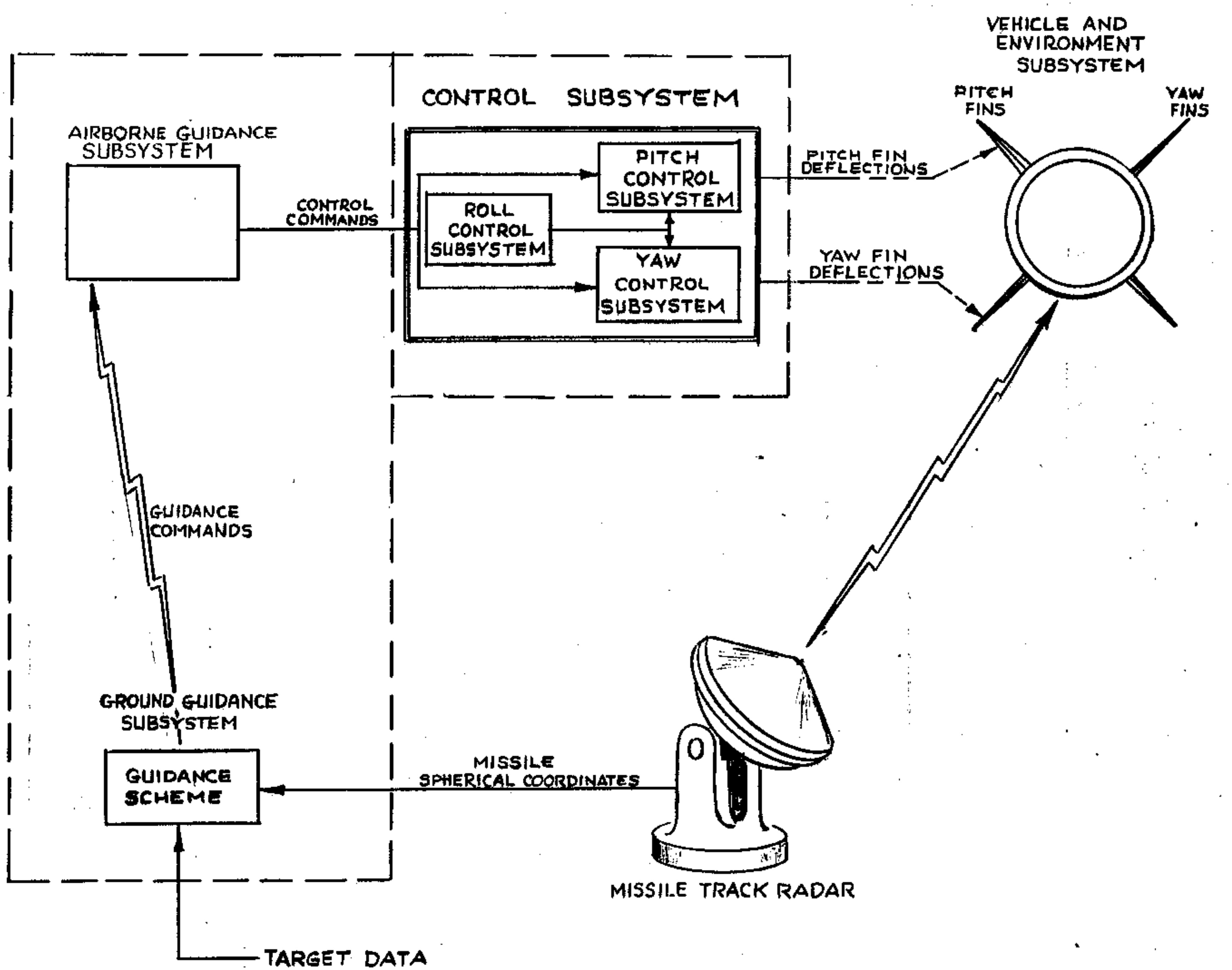


FIG 1-7 Physical System model

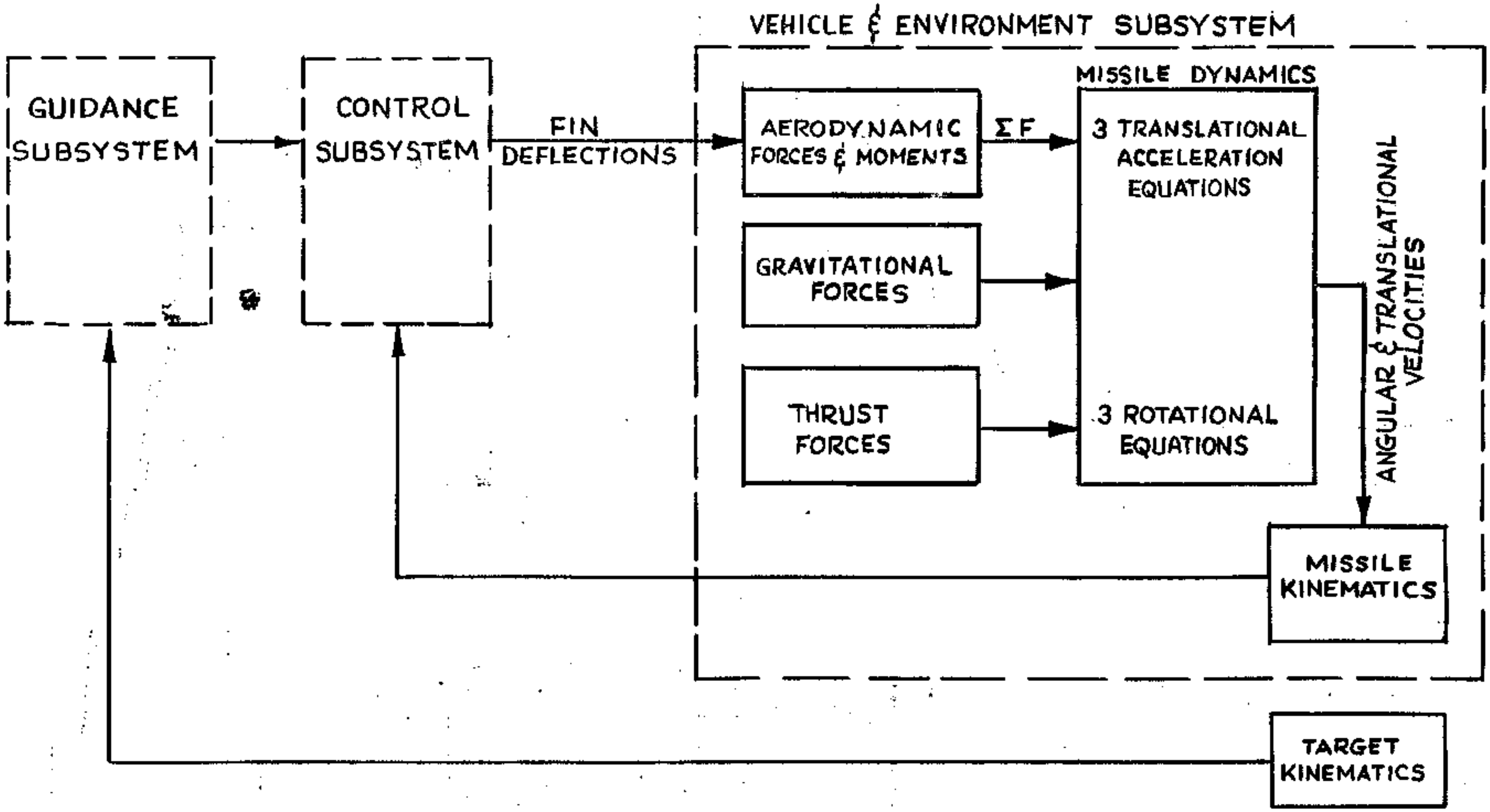


FIG. 1-8 Simulated Target Interceptor

Vehicle and Environment Subsystem

This subsystem may be broken down with regards to physical or simulated subsystem.

Physical Vehicle and Environment Subsystem

The breakdown for the physical system is:

1. Airframe.

2. Propulsion.

3. Warhead.

Simulated Vehicle and Environment Subsystem

The breakdown for the simulated system is:

1. Forces and moments on the airframe.

2. Missile dynamics.

3. Missile kinematics.

4. Warhead.

Kinematics refer to the motion of geometrical points whereas dynamics refer to the motion of masses.

1. The forces and moments to be simulated are:

   a. Aerodynamic.

   b. Thrust.

   c. Gravitational (Newtonian).

2. Rigid Body.

The missile dynamics refer to the three translated and three rotational (for a rigid body assumption) scalar equations of motion, based on the two vector equations.

$$m\bar{A}_M = \bar{F}_A + \bar{F}_G + \bar{F}_T \qquad (1\text{-}1)$$

$$\frac{d}{dt}\left(I\bar{\omega}_B\right) = \bar{L} + \bar{M} + \bar{N}\ . \qquad (1\text{-}2)$$

ANY ONE OF THE MANY EXPRESSIONS FOR THE INERTIAL ACCELERATION OF THE MISSILE OF EQUATIONS (10-24) THROUGH EQUATION (10-48) MAY BE USED IN EQUATION (1-1) ABOVE FOR THE TRANSLATIONAL ACCELERATION OF THE MISSILE.

NON-RIGID BODY

EQUATIONS DEFINING THE FLEXURE OF THE AIRFRAME (ELASTICITY) INCLUDED IN THE SIMULATION WILL REVEAL ANY INSTABILITY CAUSED BY SENSING DEVICES SENSING THE VIBRATIONS OF THE AIRFRAME.

3. MISSILE KINEMATICS.

THIS BLOCK (SEE FIG. 1-7) COVERS A NUMBER OF SETS OF EQUATIONS, DEPENDING UPON THE REFERENCE FRAME IN WHICH THE MISSILE DYNAMICS ARE SOLVED AND ON THE SUBSEQUENT USE OF THE ACCELERATION OF THE MISSILE CENTER OF MASS. SOME OF THE EQUATIONS (FOR EXAMPLE) SOLVE FOR THE BODY ATTITUDE ANGLES, FOR TRAJECTORY COORDINATES, FOR MISSILE SPEED, FLIGHT SPEED, FLIGHT PATH ANGLES, AND OTHER VARIABLES RELATING TO THE GEOMETRICAL ASPECTS OF FLIGHT.

CONTROL SUBSYSTEM

THE CONTROL OF THE ORIENTATION AND THE MOTION OF THE MISSILE IS ACHIEVED INDIRECTLY THROUGH THE CONTINUOUS CONTROL OF TWO OF THE FORCES (THRUST AND AERODYNAMIC) OF EQUATION (1-1), THAT IS:

$$M\overline{A}_M = (\overline{F}_T + \overline{F}_A) + \overline{F}_{NG}$$

WHICH DEFINES THE MOTION OF THE MISSILE.

THE THRUST FORCE $\overline{F}_T$ MAY BE CONTROLLED BY GIMBALLED MOTORS, VARIABLE THRUST (MAGNITUDE), JET VANES, JET BOTTLES ETC.

THE AERODYNAMIC FORCE $\overline{F}_A$ IS CONTROLLED (WHILE IN THE SENSIBLE ATMOSPHERE ONLY) BY CONTROLLING THE POSITION OF MOVABLE CONTROL SURFACES.

THE CONTROL-SURFACE POSITIONING SERVOS ARE USUALLY CONSIDERED FOR TWO ORTHOGONAL PLANES (LOOSELY REFERRED TO AS PITCH PLANE AND YAW PLANE). THE MAJORITY OF PRESENT DAY CONVENTIONAL MISSILE SYSTEMS MAY BE CONSIDERED TO HAVE THREE PRIMARY (AERODYNAMIC FORCE) SERVO CONTROL LOOPS:

1. PITCH PLANE CONTROL LOOP.

2. YAW PLANE CONTROL LOOP.

3. ROLL CONTROL LOOP (FOR ROLL STABILIZED MISSILES).

ANOTHER MAJOR SERVO CONTROL LOOP IS CONTROLLING THE ORIENTATION OF A GIMBALED BODY WHICH MAY BE:

1. RADAR DISH CONTROL SERVO.

2. STABLE PLATFORM CONTROL SERVO.

GUIDANCE SUBSYSTEM

A DISTINCT BREAK BETWEEN GUIDANCE AND CONTROL MAY BE NON-EXISTENT, HOWEVER, A CONVENIENT BREAK IS OFTEN MADE. EXACTLY WHERE THE BREAK OCCURS IS NOT REALLY IMPORTANT, SINCE THE MECHANISM UNDER CONSIDERATION CAN FUNCTION ONLY AS AN INTEGRATED INTERACTING SYSTEM. CONSEQUENTLY, THE BREAKDOWN MAY BE A FUNCTION OF THE INDIVIDUAL STUDY GROUPS CONCEPT OF ESTHETICAL FOR THE PARTICULAR MISSILE UNDER STUDY.

THIS REPORT CONSIDERS THE FUNCTION OF GUIDANCE AS THE "BRAINS" OF THE SYSTEM. EXAMPLES SERVE BETTER THAN DEFINITIONS FOR THIS MULTI-FACET CONCEPT. FOR EXAMPLE, CONSIDER AN AIR-TO-AIR INERTIALLY GUIDED MISSILE TO BE SLAVED TO A STRAIGHT LINE AT THE LAUNCH ALTITUDE. ONE COULD MEASURE THE ACCELERATION NORMAL TO THE VERTICAL PLANE (FLAT, NON-ROTATING EARTH ASSUMPTION) AND NULL OUT THE DOUBLE INTEGRAL (LINEAR NORMAL DISPLACEMENT OF THE MISSILE FROM THE VERTICAL PLANE) AND CONSEQUENTLY SLAVE THE MISSILE CENTER OF MASS TO THE VERTICAL PLANE. WITHIN THE VERTICAL PLANE ONE COULD SLAVE THE MISSILE TO A CONSTANT ALTITUDE. THE RESULT OF THE TWO GEOMETRICAL CONSTRAINTS ON TWO OF THE RECTANGULAR COORDINATES OF THE MISSILE LEAVES ONLY ONE DEGREE OF FREEDOM FOR THE MOTION OF THE MISSILE CENTER OF MASS.

SUCH A SYSTEM COULD CONCEIVABLY HAVE NAVIGATION ACCELEROMETERS ON A STABLE PLATFORM TO MEASURE THE ACCELERATION OF THE CENTER OF MASS NORMAL TO THE VERTICAL PLANE. AT THE SAME TIME A BODY FIXED ACCELEROMETER MAY BE A PART OF THE CONTROL LOOP TO INCREASE THE RESPONSE OF THE AIRFRAME MOTION CONTROL LOOP. THUS THE TWO SENSING DEVICES MAY BE CONSIDERED AS:

1. NAVIGATIONAL ACCELEROMETERS (HIGH ACCURACY).

2. CONTROL ACCELEROMETERS (LOW ACCURACY REQUIREMENTS).

THE GUIDANCE PHILOSOPHY FOR THE ABOVE EXAMPLE IS TO SENSE AND COMPUTE TWO COORDINATES OF THE MISSILE CENTER OF MASS AND EQUATE THEM TO ZERO. THE CONTROL FUNCTION IS TO BE SURE THAT THE MISSILE CONTROL SURFACES MOVE SO THAT THE AERODYNAMIC FORCE CAUSES THE NEEDED MISSILE ACCELERATION TO NULL OUT THE DISPLACEMENT. THUS, IF THE MISSILE IS ROLL STABILIZED THE TASK MAY BE SIMPLE, HOWEVER, IF THE MISSILE IS SPINNING, THE COORDINATION OF FIN MOTION IS COMPLICATED.

THE GUIDANCE ALSO COMPUTES WHEN TO SEPARATE STAGES AND VARY THRUST. ONE OF THE MAJOR FUNCTIONS OF THE GUIDANCE SYSTEM IS THE MISSILE-TARGET KINEMATICS. IN THE EXAMPLE OF THE STRAIGHT LINE TRAJECTORY THE TARGET WAS ASSUMED TO BE ON THE LINE. THUS, A BREAKDOWN IN THE FUNCTIONS OF THE GUIDANCE SYSTEM MAY BE:

1. MISSILE KINEMATICS (SENSING MISSILE GEOMETRY VARIABLES AND THEIR

TIME OF RATES OF CHANGE).

2. MISSILE-TARGET KINEMATICS.

3. GUIDANCE SCHEME (FEEDS INFORMATION TO CONTROL SUBSYSTEM DEPENDING ON PARTICULAR GUIDANCE PHILOSOPHY).

WARHEAD SIMULATION

THE GUIDED MISSILE WILL STATISTICALLY DELIVER THE MISSILE CENTER OF MASS TO A DESIGNATED POINT IN SPACE. THE CIRCULAR PROBABLE ERROR (CEP) - SPHERE OF A GIVEN RADIUS INTO WHICH 50 PERCENT OF THE MISSILES WILL BE DELIVERED IS A MEASURE OF THE ACCURACY OF THE GUIDED MISSILE SYSTEM. ONE MUST SUPERIMPOSE THE STATISTICAL EFFECTS REFLECTING THE LETHALITY OF THE WARHEAD UPON THE GEOMETRICAL PROBLEM OF MASS DELIVERY (GUIDED MISSILE ACCURACY).

ENVIRONMENT

THE SIMULATION OF EXTERNAL WINDS, ATMOSPHERIC CONDITIONS, GRAVITATIONAL ANOMALIES, HARDWARE RESPONSES UNDER VIBRATION, RADIATION FIELD, TEMPERATURE ETC., MUST ALSO BE INCLUDED IN THE STUDY.

TARGET MAJOR SUBSYSTEM

THE FUTURE STATE-OF-THE-ART OF ENEMY HARDWARE DEVELOPMENT IS THE PRIMARY FACTOR IN DEFENSIVE WEAPON SYSTEM DESIGN REQUIREMENTS. THUS, IN A STUDY OF THE TYPES I THROUGH IV OF FIG. 1-3, ONE MAY SIMULATE TARGET MOTION.

THIS SIMULATION MAY ENTAIL A COMPLETE GUIDED MISSILE SIMULATION E.G. AN ICBM RE-ENTERING THE ATMOSPHERE ETC., TO OBTAIN MORE SIMPLE MATHEMATICAL EXPRESSIONS DEFINING THE GEOMETRICAL BEHAVIOR OF THE TARGET.

WEAPON SYSTEM MATHEMATICAL MODEL

A FEW OF THE MANY VALUES TO BE DERIVED FROM THE MATHEMATICAL MODEL ARE:

1. GIVE TO ALL INTERESTED PERSONNEL AN UNDERSTANDING OF THE BASIC SYSTEM THEORETICAL DESIGN EQUATIONS. THIS FUNCTION SHOULD SEPARATE THE GUIDANCE AND CONTROL PHILOSOPHIES FROM PARTICULAR INSTRUMENTS USED IN THE PHYSICAL REALIZATION OF THE THEORETICAL EQUATIONS.

2. REVEALS WHERE THE DESIGNER OF THE SYSTEM HAS MADE APPROXIMATIONS AND ASSUMPTIONS FOR THE RELAXATION OF INSTRUMENTATION REQUIREMENTS, THUS, POINTING OUT AREAS OF INVESTIGATION OF THE DEGRADATION OF THE SYSTEM PERFORMANCE DUE TO THIS ECONOMY OF EQUIPMENT.

3. THE MATHEMATICAL MODEL OF THE SYSTEM IS IN A LANGUAGE UNDERSTANDABLE TO ELECTRONIC COMPUTERS, THUS ENABLING VERY INVOLVED SYSTEM STUDIES TO BE MADE.

Criterion For The Selecting of a Mathematical Model For Simulation

The selection of an "optimum" system of equations defining the physical system under consideration is a relatively difficult task. Many of the equations defining the subsystems may be obtained by conventional analysis techniques. For example, the lumped-parameter assumption utilized in deriving the equations for passive networks in the control subsystem. However, the simulation of the airframe orientation, the axes along which to solve the missile translational motion, the simulation of the various other reference frame orientations, the method of generating the corresponding direction cosines as functions of the various Euler sequences, may be done in many ways. The answers to the above problems are not unique and the choices should be based on a consideration of the over-all system. Some of the criterion in the selection of the mathematical model are:

1. The mathematical model should contain the system variables which can and will be measured during the actual missile flights for comparison.

2. Simplicity in form and number of equations.

3. Compatibility of simulation mathematical model equations to the type of computer being used whether digital or analog.

The importance of the compatibility of system data measurements and mathematical model variables is borne out by a statement from Goode and Machol.[28]

"Almost invariably operational experiments and tests on large-scale systems cost more money and take more time than was originally estimated; almost invariably it appears afterward that more effort could profitably have been spent on planning before the measurements were undertaken; and almost invariably the task of data reduction is underestimated. If a dozen motion picture cameras are recording as many objects (scopes, dial readings, and the like) at the usual rate of 24 frames per seond, then a million frames are being accumulated per hour. If the data reduction involves manual examination of the film by frame as it often does, then most of the film is probably destined for dead storage without ever being used."

Also, the telemetry and other data gathering requirements of the range must be a function of the system evaluation criterion requirements, thus the intimate link between the mathematical model and the feasible experiments must be considered.

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[28] Ibid.

AS ANOTHER EXAMPLE, THE MISSILE TRANSLATIONAL ACCELERATION EQUATIONS OVER A ROTATING SPHEROIDAL EARTH MAY APPEAR VERY SIMPLE IN FORM IF THEY ARE SOLVED IN AN INERTIAL REFERENCE FRAME, AND THE COMPONENTS OF THE AERODYNAMIC, GRAVITATIONAL, AND THRUST FORCES ARE TRANSFORMED TO THE INERTIAL FRAME. HOWEVER, IF THE MISSILE HAD A PLUMB-BOB STABLE PLATFORM ABOARD AND THE ATTITUDE CONTROL WERE BASED ON THE CONTROL OF THE ORIENTATION OF THE AIRFRAME WITH RESPECT TO THE LOCAL TANGENT PLANE TO THE SPHEROID, THEN ONE WOULD LIKE TO SIMULATE THESE PLATFORM GIMBAL PICK-OFF ANGLES.

IN THE FIRST CASE ONE MAY SELECT THE EULER ANGLES ORIENTING THE BODY FRAME WITH RESPECT TO AN INERTIAL FRAME, IN THE LATTER CASE ONE WOULD BE REQUIRED TO ORIENT THE BODY FRAME WITH RESPECT TO THE PLATFORM FRAME (HAVING) ONE VECTOR ALONG THE LOCAL PLUMB-BOB VERTICAL AND THE OTHER TWO VECTORS SOMEWHERE IN THE TANGENT PLANE DEPENDING ON THE TYPE OF PLATFORM AZIMUTH SLAVING DESIGNED INTO THE SYSTEM.

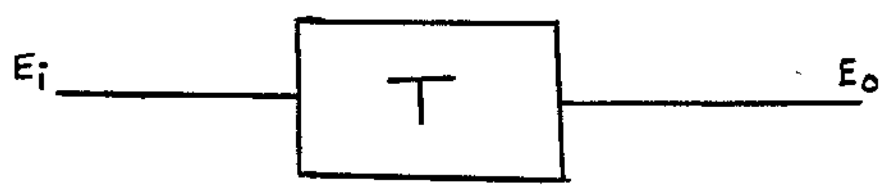
CONSEQUENTLY, FOR LARGE SCALE SYSTEM SIMULATION ONE SHOULD NOT ARBITRARILY SELECT COORDINATE SYSTEMS BASED ON SIMPLICITY OF EQUATIONS ALONE.

THE SELECTION OF THE SIMULATION COORDINATES SHOULD BE ALSO BASED ON THE GUIDANCE AND CONTROL SENSING DEVICES OF THE PHYSICAL SYSTEM WHICH SENSES THE COORDINATES AND THEIR TIME RATES.

IN A SIMILAR MANNER, ONE SHOULD CONSIDER THE SELECTION OF EQUATIONS THROUGHOUT THE SYSTEM.

THE OBSERVATION OF PHYSICAL PHENOMENA IN ITS ENVIRONMENT IS THE FIRST REQUIREMENT OF AN "INTELLIGENT" WEAPON SYSTEM. SINCE THERE APPEARS TO BE NO PHYSICALLY REALIZABLE OMNI-OBSERVATION DEVICES, THE PHYSICAL PROPERTIES OF THE PHENOMENA MUST BE OBSERVED SELECTIVELY. CONSEQUENTLY, THE INSTRUMENT MAKING THE OBSERVATION IS SENSITIVE PIECEWISE. THUS, A FUNDAMENTAL CONCEPT TO ALL PHYSICAL SCIENCES UTILIZING MATHEMATICAL ANALOGIES IS THE CONCEPT OF TRANSFER RELATION OR TRANSFORMATION.

CONSIDER SOME PHENOMENA E_i OF INTEREST TO BE OBSERVED BY A DEVICE HAVING KNOWN BEHAVIOR DESCRIBED MATHEMATICALLY BY T AND THE OUTPUT E_o KNOWN,



THEN THE PROBLEM HERE IS TO DETERMINE WHAT E_i REALLY IS. IN GENERAL, A DEVICE WILL BE SENSITIVE TO PHYSICAL PHENOMENA OTHER THAN THAT OF INTEREST. FOR EXAMPLE, IT MAY BE DESIRED TO MEASURE THE ACCELERATIONS OF A VEHICLE WITH RESPECT TO INERTIAL SPACE BY USING AN AIRBORNE INERTIAL ACCELEROMETER, SINCE THE PROOF-MASS IN THE ACCELEROMETER IS ITSELF SENSITIVE TO THE NEWTONIAN MASS ATTRACTION FIELD, THE INSTRUMENT IS INCAPABLE OF MEASURING THAT COMPONENT OF INERTIAL ACCELERATION OF THE CARRYING VEHICLE CAUSED BY THE GRAVITATIONAL FORCE ON THE VEHICLE-MASS.

Any one of the three quantities E_i, E_o, T, may be considered as the unknown. If E_i is considered as target motion and E_o is a specified target interceptor path, then the problem is to design a system T to deliver the interceptor sufficiently near the target, a problem of system synthesis (design).

The problem of determining what the output of a given system T will be in the presence of specified inputs E_i is the problem of analysis of T.

The transformation T may be linear or non-linear, and the inputs and outputs may be continuous or discrete, that is the device may be Analog and Digital or a combination of devices containing many sub-loops, some of which are Analog in nature and some of which are Digital in nature.

Thus, a mathematical description of the geometrical relations and the system of constraint equations which effect it are the basis from which "hardware" (sensing, control, thrust, etc., devices) may be designed to physically realize the desired situation.

Weapon systems may be considered as Defensive W. S. or as Offensive W. S. In both cases, fundamentally one is concerned with sensing the space coordinates of a target and a target interceptor and translating the coordinates of the interceptor sufficiently close to the target coordinates.

The process of geometrical variable sensing requires energy either radiated or reflected. Kinetic energy is required for dynamic inertial sensing utilizing accelerometers and gyros. Electro-magnetic energy is required in optical tracking, radar tracking, and infra-red tracking systems. Acoustic energy is involved in acoustic trackers. Thus the concept of geometrical <u>variable sensing systems</u> may be classified in the following way.

<u>Internal Sensing Systems</u>. These include accelerometer, gyros, stable platforms (generally utilizing gyros in the loop), etc.

<u>External Sensing Systems</u>. This class of sensing systems refers to those having some of the sensing loop system equipment external to the intercepting missile (a radiating star is not considered as system equipment, whereas a target acting as an energy reflector is a part of the sensing equipment loop). Consequently this class of devices includes the familiar radar, infra-red trackers and acoustic trackers.

A sharp distinction between external and internal geometrical sensing devices is not important, the gist of the classification is to accentuate the geometrical aspects of weapon system design.

Sensing systems may also be classified with respect to the location of the energy devices, receiver or transmitter. These well known classifications are:

1. <u>Fully Active</u>. In an active system the energy source (transmitter) and the receiver are in the intercepting missile.

2. <u>Semi-Active</u>. In semi-active systems the energy sensing device is located in the intercepting missile and senses reflected energy from the target after having been illuminated from the energy source which is a system device located external to the missile.

3. <u>Passive</u>. In a passive sensing system, the energy source is not a portion of the designed system, however the receiver is generally located in the missile. Thus a star-tracker may be considered as a passive sensing device just as is an infra-red seeker locked onto a target's heat.

The measurement of land-fixed target coordinates for a surface-to-surface missile may have been measured on the time scale many years back. The measurement of coordinates for an air-to-surface missile may be optical or infra-red devices. The measurement of the coordinates of a moving target for an air-to-air interceptor say, may be much more recent data. In all cases, accuracy of measuring devices are a factor, whether survey instruments or high precision radar.

The design of devices to culminate the sufficient proximity of the interceptor and target coordinates is the prime objective of a weapon system design.

Fundamental to the idea of weapon system evaluation is a measure of effectiveness. One factor to be considered in such a measure is the kill probability.

The probability of kill of a missile can be expressed as a probabilistic function of three factors:

1. Probability of airframe delivery for a design C.E.P. for a properly functioning missile system.

2. Reliability of missile system.

3. Probability of kill by a missile which is delivered to the design C.E.P.

The second factor can be determined only after production begins, for it is a function of hardware design and fabrication and is determined empirically, however, reliability (pre-hardware) factors may be considered prior to production.

The two main factors which effect the reliability are:

1. The state-of-the-art of hardware design and fabrication.

2. The adverse effects of the environment.

The block diagram of Figure 1-9 shows the relation:

Error analysis through simulation is an aid to economical system reliability design. Such simulation studies will eliminate the bad design practice of the past of specifying extreme component accuracy and manufacturing precision, simply because an adequate error analysis study has not been performed stating where precision is and is not required. These extreme accuracies are reflected in equipment reliability.

Simulation can be used to determine reliability requirements of components. The mathematical model of a subsystem may be simulated and the values of the component parameters changed to study their effects on subsystem input-output relation.

For example the mathematical model may take into account the correlation between circuit performance parameters, (gain, bandwidth, power output etc.) and can be used with any form of probability density function for the "behavior statistics" of the components (statistical distribution of the component characteristics as a function of time and environment).

Simulation can also be used to determine the reliability requirements of subsystems in series, parallel, and series-parallel arrangements.

Bellman and Dreyfus,[29] of the Rand Corporation, used a digital computer to simulate the mathematical model of a parallel redundant system to determine the most efficient duplication procedure.

Lipp,[30] of the General Electric Company, in his paper "Topology of Switching Elements vs Reliability" disproves the belief that circuit complexity necessarily lessens reliability and further develops a simple mathematical model adaptable to the solution of complex circuits in terms of reliability.

The analytical objectives of a system reliability program are two-fold:

1. The specified system reliability requirements must be assigned to the subsystems and components to assure adequate system design.

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[29] By permission from Dynamic Programming and Reliability of Multicomponent Devices by R. Bellman and S. Dreyfus. Copyrighted 1958. The Journal of the Operations Society of America, Vol. 6, Nr 2, Mar-Apr 1958.

[30] By permission from The Topology of Switching Elements vs Reliability by J. P. Lipp. Copyrighted 1957. Institute of Radio Engineers, June 1957.

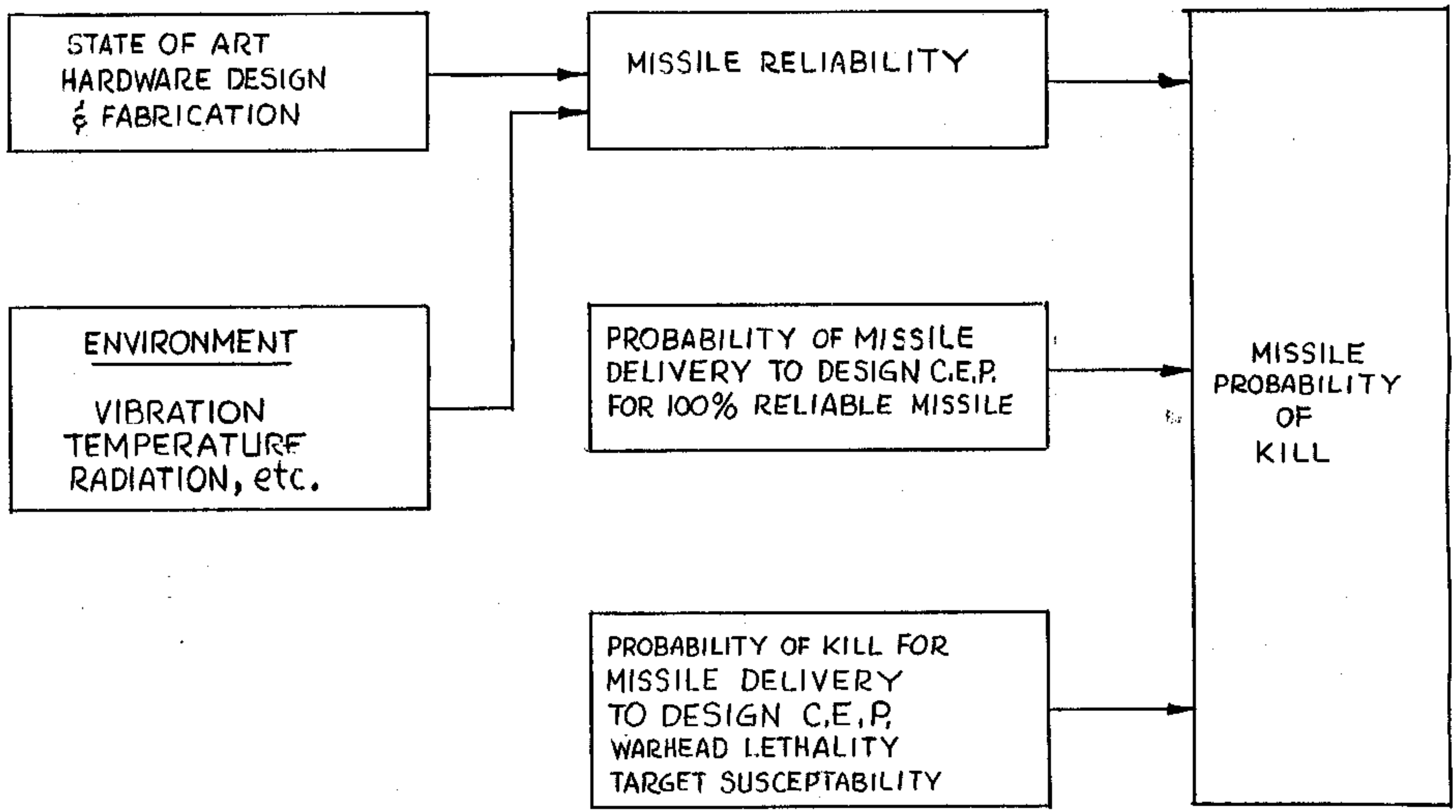

FIGURE 1-9 KILL PROBABILITY FACTORS
75

2. As component and subsystem design data are made available throughout the development and testing program for various environmental conditions, this information must be utilized in predicting the system reliability. Youtcheff,[31] states: "In predicting system reliability, it is necessary to determine both component reliability relationships and the individual component failure probabilities. Several statistical methods for determining component reliability are presented; however, the exact methodology must be tailored to the specific system and development program. System reliability can be predicted by utilizing component reliability data together with an adequate analysis of component and subsystem reliability relationships."

Stevens,[32] in his paper, "A Sequential Test For Comparing Component Reliabilities" states: "If a digital computer is available, however, it is feasible to simulate the test process a large number of times by a 'Monte Carlo' method, thus obtaining any information required about the mode of operation of the tests."

A program requiring the Contracting Agency (Government) to evaluate a weapon system concurrently with the developing contractor is a natural step in the unified national effort to maintain a lead in the conquest of effective weapon systems and space technology. The above hypothesis is a consequence of the ever increasing complexity of the systems with the resulting expensive and time consuming requirements for engineering evaluation. Accepting as inevitable the nonfeasibility of two successive independent evaluation programs for our advanced weapon systems this section discusses some pertinent aspects of a concurrent flight evaluation program.

The task of evaluating a missile system based primarily on data obtained from the contractor or contractor-government firing experiments requires a well planned program utilizing to the fullest extent automation concepts. At a very early stage of system development a scientific team should develop the system mathematical model. This mathematical model should contain the system variables which can and will be measured in subsequent tests in the laboratory and on the range.

Present range instrumentation system and data reduction describes the missile flight and internal parameter behavior but does not evaluate it. This part considers the problem of evaluating missile flight tests by a

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[31] By permission from <u>Statistical Aspects of Reliability in Systems Development</u> by J. S. Youtcheff. Copyrighted 1957. Transactions on Reliability and Quality Control, Nov 1957.

[32] By permission from <u>A Sequential Test For Comparing Component Reliabilities</u> by C. F. Stevens. Copyrighted 1957. Institute of Radio Engineers Transactions on Reliability and Quality Control, November .

COMBINATION OF A STUDY OF THE PHYSICS OF THE SYSTEM AND EXPERIMENTAL TEST DATA. THE ASSUMPTION IS MADE THAT A COMPLEX WEAPON SYSTEM CANNOT BE FULLY UNDERSTOOD UNTIL A COMPLETE MATHEMATICAL FORMULATION OF THE PHYSICALLY REALIZED SYSTEM HAS BEEN ACCOMPLISHED AND THE RESULTING EQUATIONS SOLVED A SUFFICIENT NUMBER OF TIMES TO THOROUGHLY DEPICT THE SYSTEM BEHAVIOR OVER THE AREAS OF INTEREST.

IN CASES WHERE THE EQUATIONS ARE TOO DIFFICULT TO OBTAIN THE SUBSYSTEM MAY BE TIED INTO THE PRE-FLIGHT SIMULATION. THESE PRE-LAUNCH SIMULATED FLIGHTS WOULD MAKE KNOWN THE EXPECTED BEHAVIOR OF MANY OF THE SYSTEM VARIABLES DURING THE LIVE-FLIGHT, THUS ENABLING A BETTER UNDERSTANDING OF THE ACTUAL FLIGHT AS WELL AS SERVING AS A GUIDE IN DETERMINING WHAT THE FLIGHT SHALL TEST. AFTER THE SIMULATED FLIGHT OF THE SUBSYSTEM IN THE LABORATORY IS COMPLETED THE MUCH MORE EXPENSIVE FLIGHT TEST OF THE SUBSYSTEM MAY BE PERFORMED. IN THIS MANNER A MINIMUM OF INCREASINGLY-DEMANDED RANGE TIME IS CONSUMED THROUGHOUT THE DEVELOPMENT OF THE SYSTEM.

THE PRIMARY OBJECTIVE OF THE EVALUATION OF THE MISSILE TEST FIRING DATA IS THE MEASUREMENT AND INTERPRETATION OF EXTERNAL PERFORMANCE CAPABILITIES (RANGE, VELOCITIES, MANEUVER CAPABILITIES, ACCURACIES) AS DETERMINED BY THE BASIC SYSTEM PARAMETERS SUCH AS GAIN SETTINGS, AIRFRAME CONFIGURATION CHANGE, THRUST, ETC. CONSEQUENTLY, IT IS NECESSARY TO MEASURE MANY OF THE INTERNAL MISSILE FUNCTIONS DURING FLIGHT FOR PURPOSES OF FAILURE DIAGNOSIS AS WELL AS ANALYSIS OF FACTORS GOVERNING VARIATION IN PERFORMANCE.

THE IN-FLIGHT BEHAVIOR OF THE ELEMENT OR SUBSYSTEM UNDER INVESTIGATION IS REPRODUCED BY POST SIMULATION; THUS IDENTIFYING ITS PARAMETER CONFIGURATION. BY THIS MEANS CONSIDERABLE INSIGHT INTO CAUSES AND EFFECTS OF IN-FLIGHT FAILURES CAN BE OBTAINED AS WELL AS A CLARIFICATION OF THE RELATION BETWEEN INTERNAL PARAMETERS AND OVERALL PERFORMANCE.

IN ORDER TO CHECK THE VALIDITY OF THE SYSTEM OF EQUATIONS, A CRITERION MUST BE ESTABLISHED TO MEASURE THE FIGURE OF MERIT OF THE SIMULATION. THE DRIVING FUNCTIONS TO THE POST SIMULATED-FLIGHT SHOULD BE TELEMETERED PLAY BACK DATA. THE SIMULATED FLIGHT IS THEN CONDUCTED IN A MANNER SIMILAR TO THE ACTUAL TEST WHOSE RESULTS ARE BEING ANALYZED. IDEALLY, FOR THE SAME INPUTS TO THE SIMULATED FLIGHTS AS FOR THE REAL FLIGHT, IT IS DESIRED TO OBTAIN THE SAME OUTPUT AS A FUNCTION OF TIME. PRACTICALLY THIS CANNOT BE ACHIEVED AND IT IS NECESSARY TO DEVELOP THE ABOVE CRITERIA. THESE EVALUATION CRITERIA SHOULD BE MECHANIZED ON COMPUTERS SO THAT FLIGHT EVALUATION IS SPEEDED UP IN ORDER TO INFLUENCE THE SUCCEEDING FLIGHT TEST. GREAT SIMILARITY BETWEEN THE SIMULATED FLIGHT AND THE ACTUAL FLIGHT VARIABLES INDICATES THAT THE ACTUAL FLIGHT TEST VEHICLE PERFORMED IN A MANNER ESSENTIALLY SIMILAR TO A UNIT WITH TYPICAL OR NOMINAL RESPONSE. LIKEWISE, WIDE DIFFERENCES BETWEEN THE SIMULATED AND ACTUAL FLIGHT TEND TO INDICATE THAT THE UNIT HAD FUNCTIONED IN SOME IRREGULAR MANNER AND TO PROVIDE AN ESTIMATE OF THE NATURE AND MAGNITUDE OF THE IRREGULARITY, FOR EXAMPLE, FROM AN OBSERVATION OF THE TELEMETERED DATA ALONE, A SPURIOUS LARGE TRANSIENT IN THE TELEMETRY LINK MAY LOOK THE SAME AS A TRANSIENT WHICH OCCURRED IN THE MISSILE COMMAND LINK.

Failures may be located depending on a manner of circumstances some of which are: Whether the abnormal behavior can be reproduced on the post simulated flights; whether the response characteristics of the system for various states of the internal system parameters can be shown to be unique; whether the degree to which the effects of the responses of individual elements or stages can be isolated.

Pre-flight simulations utilizing flight-tables system hardware, etc., could reveal the effects of reference voltage level variations, bottoming of instruments, total component malfunction, parameter gain setting deviations, under simulated dynamic conditions. Actual flights are far too expensive to permit investigating these many possible internal deviations. As a consequence the live-flight tests are "ideal set-up" experiments, for example, in the present lengthy pre-launch check-out time, all system gain settings are painstakingly adjusted to give optimum performance.

The most economical evaluation program can be effected through simulation by determining the factors which give rise to the statistical distribution of performance characteristics, estimating the mean values and dispersions, and verifying conclusions by means of a minimum number of actual firing tests, by injecting signals having "typical" noise spectrums, and by a careful analysis of factors leading to dispersions of the system variables, the statistics of system performance may be inferred.

Although there are many many more degrees of freedom in the actual system than in the mathematical model, pre-flight study would select the critical and significant system variables to be monitored during the flight. Since the mathematical model of the physically realized system is only limited by the accuracy of the experimental data, the above study would indicate accuracy requirements of the measuring devices.

Computers are the basic tools for implementing in a timely manner the proposed evaluation program. The present state-of-the-art of simulation using high speed digital and analog machines necessitates a simulation set-up for each active missile under development. The extensive pre-launch and post-launch simulation of a particular missile system would require continuous simulated flights between firings to most effectively utilize the flight test data.

In conclusion, one may consider any complex system as a collection of $K_1 \cdot K_2 K_3 \ldots \times 10^N$ individual components. Thus the most simple electromechanical system would have $10^0 = 1$ components. Consider the system as being a resistor. Depending upon the degree of approximation desired, one could develop the mathematical model and a simulator to solve the equations. As a first approximation, laboratory experiments would yield a linear relation between the voltage across the resistor e and the current i, that is:

$$e = Ri. \tag{1-3}$$

One could solve the above expression by means of paper, pencil and his head as a computer, or any other scaled linear device which would be the analog of the physical resistor.

If one wanted a more realistic mathematical model the effect of temperature could be considered. The resistance of a conductor depends upon the temperature as well as the material and form, the factors which determine the value of the parameter R above. To a first approximation, the change in resistance produced by a variation in temperature is proportional to the temperature temperature increment, provided this increment is not too large. Hence the relation between the resistances R_{T_1} and R_{T_2} of a conductor at two temperatures T_1 and T_2, respectively, is:

$$R_{T_2} = R_{T_1} \left[1 + \kappa \left(T_2 - T_1\right)\right] \qquad (1\text{-}4)$$

where κ is the fractional change in resistance per degree of temperature rise, and is termed the temperature coefficient of resistance (values depending upon the type of material).

Once the values of K and R_T have been determined for a given material, Eq. (1-4) can be utilized in Eq. (1-3)

$$\varepsilon = R_{T_1} \left[1 + \kappa \left(T_2 - T_1\right)\right] i \;. \qquad (1\text{-}5)$$

The solution of Eq. (1-5) is now more complicated than Eq. (1-3). If the effects of humidity H are considered and the lumped parameter expression of Eq. (1-3) is replaced by a distributed parameter assumption, that is the inductive and capacitive effects of the wire windings are considered as a function of the frequency ω of ε, one obtains:

$$\varepsilon = R\left(\omega, T, H \ldots\right) i. \qquad (1\text{-}6)$$

If one next attempts to streamline the testing operation, Equation (1-6) could be mechanized on a computer and the physical component subjected to environmental tests. The outputs could be compared on a real time basis as shown in Fig. 1-10.

One of the major problems of scientifically designing a complex system is to obtain the mathematical model of the physically realized system. Such a realistic model would reflect the stochastic nature of the physical device. The overall complex system modeling scheme is shown in Fig. 1-11.

Conceding to the desirability of real time flight analysis with its requisite mathematical models, one logically next considers how to utilize the results of the real-time analysis to influence (feed-back approach) the behavior of the system. Thus through real-time analysis concepts, one is directed towards the design of more sophisticated systems. These sophisticated systems are under study by the air force, National Aeronautics and

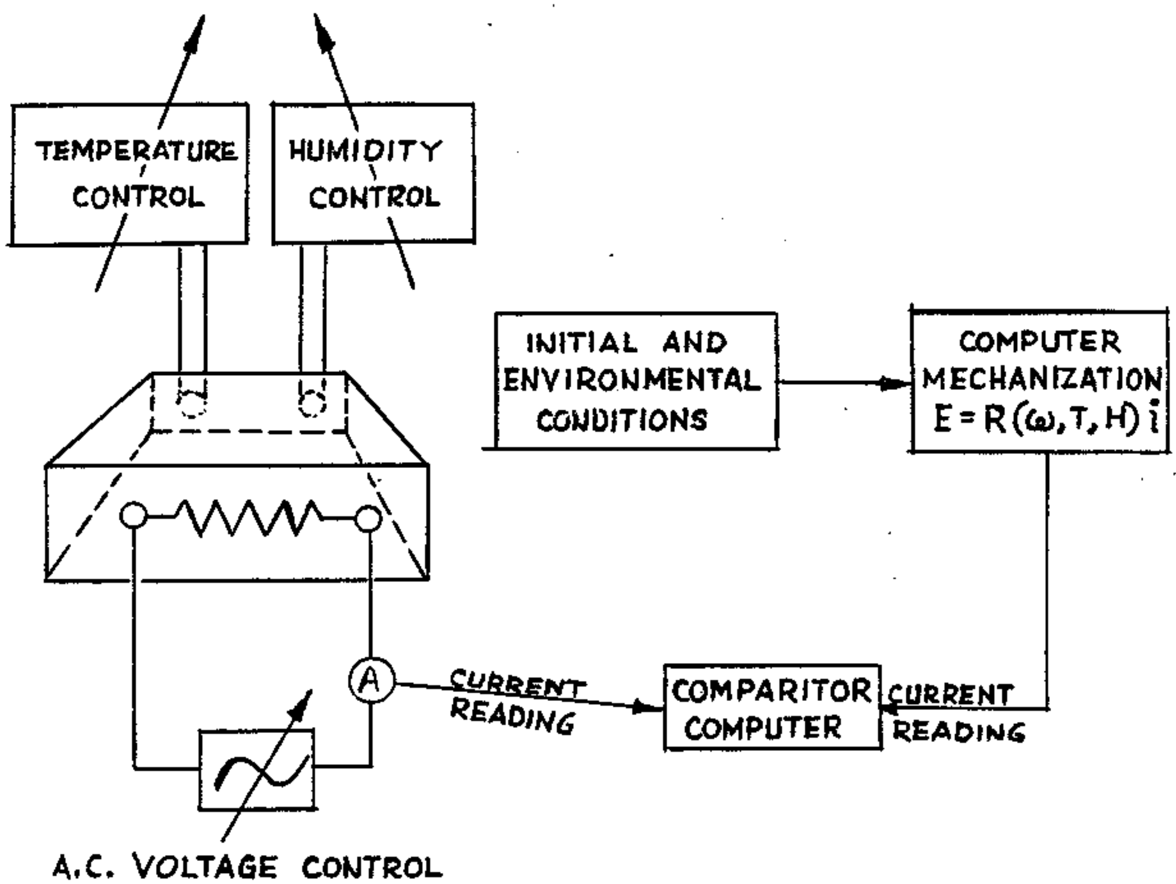
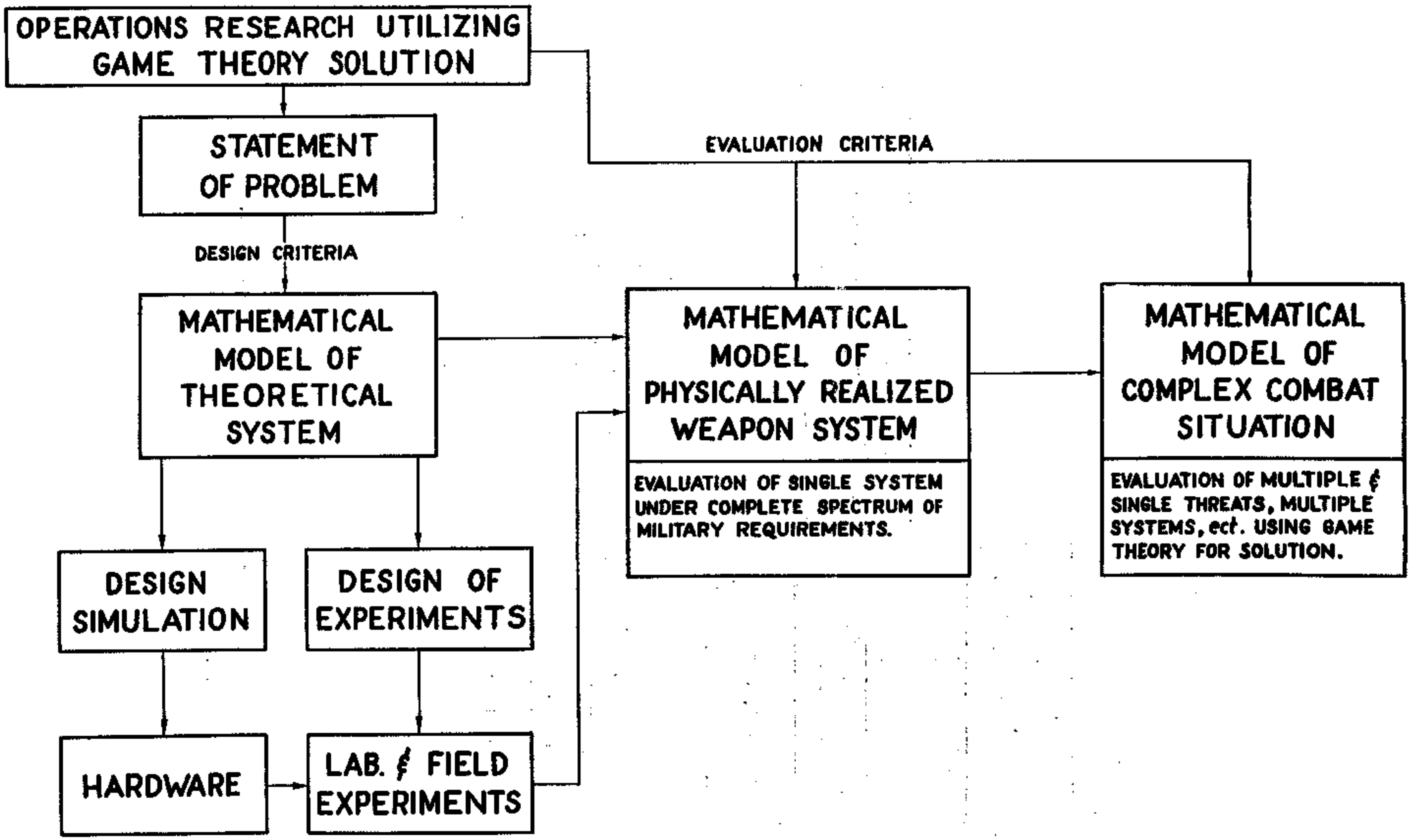


Fig. 1-10 - Real Time Analysis Scheme Elementary



THE MODEL APPROACH TO SYSTEM DESIGN AND EVALUATION

FIGURE 1-11

Space Agency, M.I.T., Sperry, Minneapolis Honeywell, and many other companies and Universities under the heading of "Adaptive Flight Control Systems." For further literature on these future sophisticated systems one is referred to the following references:

Design of Model-Reference Adaptive Control Systems For Aircraft, H. P. Whitaker; J. Yamron, A Kezer, Sept 1958; Report R-164.

A Study To Determine An Automatic Flight Control Configuration To Provide A Stability Augmentation Capability For A High-Performance Supersonic Aircraft, Minneapolis-Honeywell Regulator Co., Aeronautical Division, May 1958; W.A.D.C. Technical Report 57-349.

Final Technical Report Feasibility Study Automatic Optimizing Stabilization System; Part 1, Sperry Gyroscope Co., WSDC Technical Report 58-243 Astra Document Nr AD 155576.

PART II

GENERAL ANGULAR RELATIONSHIPS

THE ANGULAR RELATIONSHIPS BETWEEN VARIOUS REFERENCE FRAMES ASSOCIATED WITH WEAPON SYSTEMS IS ONE OF THE BASIC REQUIRED STUDIES NECESSARY IN SYSTEM SYNTHESIS AND SYSTEM ANALYSIS. FOR, IN THE DESIGN OF A WEAPON SYSTEM WHEREIN A GUIDED MISSILE IS TO BE AUTOMATICALLY VECTORED TO A TARGET, INNUMERABLE VECTOR RESOLUTIONS ARE REQUIRED. SECTIONS TWO THROUGH EIGHT IS AN ATTEMPT TO DERIVE AND COMPLY THE BULK OF THE COORDINATE TRANSFORMATIONS NECESSARY TO RESOLVE THE DYNAMIC AND KINEMATIC EQUATIONS TO EFFECT TARGET INTERCEPTION FOR A NUMBER OF CASES.

FOR EXAMPLE, A DOPPLER-INERTIAL GUIDANCE SYSTEM REQUIRES TRANSFORMATIONS OF DOPPLER VELOCITY ALONG CARRIER AXES TO COMPONENTS OF DOPPLER VELOCITY ALONG PLATFORM AXES. ANGULAR RELATIONS ARE NECESSARY FOR THE ALIGNMENT OF SLAVED PLATFORMS TO A MASTER PLATFORM FOR AIR-LAUNCHED BALLISTIC MISSILES, AND FOR THE STABILIZATION OF RADAR, CAMERAS, STAR-TRACKER, OR GIMBALLED ENGINE MOUNTS. KNOWLEDGE OF THE MISSILE ATTITUDE (HEADING, ELEVATION ANGLE, OR OTHER ANGLES DEPENDING UPON THE TYPE OF FLIGHTPATH CONTROL USED) CAN BE OBTAINED FROM GYROS AND STABLE PLATFORMS. FURTHERMORE, IN SYSTEM SIMULATION STUDIES, VARIOUS TYPES OF COORDINATE TRANSFORMATIONS ARE NECESSARY, SOME OF WHICH IN ACTUAL FLIGHT ARE ACHIEVED PHYSICALLY.

CONSEQUENTLY, ONCE A WEAPON SYSTEM GUIDANCE AND CONTROL PHILOSOPHY HAS BEEN ESTABLISHED, THE DIRECTION COSINE MATRICES OF SECTIONS TWO THROUGH EIGHT AND THE METHODS OF OBTAINING THESE DIRECTION COSINES SHOULD BE AN INVALUABLE EXPEDIENT IN THE ACTUAL MECHANIZATION OF THESE CONCEPTS INTO A PHYSICAL SYSTEM.

DIRECTION COSINE MATRICES AS FUNCTIONS OF EULER ANGLES (SUCCESSIVE AND REPETITIVE)

ONE REFERENCE FRAME MAY BE UNIQUELY ORIENTED WITH RESPECT TO A SECOND REFERENCE FRAME THROUGH THREE ORDERED ANGLES. THE EULER ANGLES FREQUENTLY ENCOUNTERED ARE OF TWO TYPES:

(1) REPETITIVE EULER ANGLES AS USED IN CLASSICAL MECHANICS IN WHICH ONE OF THE THREE ANGULAR ROTATIONS IS REPEATED AND A LINE OF NODES IS ESTABLISHED, E. G. THE THREE ORDERED ROTATIONS MIGHT BE YAW, ROLL, YAW.

(2) SUCCESSIVE EULER ANGLES IN WHICH NONE OF THE ROTATIONS ARE REPEATED, E. G. YAW, PITCH AND ROLL. THE FIRST ROTATION IS ABOUT AN AXIS OF THE ONE (INITIAL) REFERENCE FRAME, THE THIRD OF THE ORDERED ROTATIONS IS ABOUT AN AXIS OF THE OTHER (FINAL) REFERENCE FRAME, AND THE SECOND ROTATION IS ABOUT AN AXIS NORMAL TO THE FIRST AND THIRD ROTATION AXES.

THE ROLL AXIS AS CONCEIVED IN THIS REPORT IS THE $\overline{F}_1$ OR $\overline{R}_{1_i}$ (ROLL AXIS AFTER iTH ROTATION VECTOR). FOR EXAMPLE IF THE FIRST ROTATION IS A ROLL,

THEN THE ROLL AXIS IS $\overline{F}_1$, IF THE FIRST ROTATION IS EITHER PITCH OR YAW AND THE SECOND ROTATION ROLL, THEN $\overline{R}_{11}$ IS THE ROLL AXIS, THAT IS, THE ROLL AXIS AFTER THE FIRST ROTATION. IF THE LAST ROTATION IS ROLL THEN THE ROLL AXIS IS $\overline{R}_{12} = \overline{R}_1$ (FINAL POSITION OF ROLL AXIS). SIMILAR CONSIDERATIONS HOLD FOR THE PITCH AXIS AND YAW AXIS.

THUS IT IS SEEN THAT THE ANGLES ϕ (ROLL), Θ (PITCH), ψ (YAW) RESPECTIVELY ARE MEASURED IN "VARIABLE PLANES" PERPENDICULAR TO THEIR AXES OF ROTATION, WHERE THE AXIS OF ROTATION IS A FUNCTION OF THE EULER SEQUENCE USED.

GIMBALLED BODY (E. G. GYROS, STABLE PLATFORMS, AND RADAR ANTENNAS) PICK-OFF ANGLES ARE OF THE SUCCESSIVE TYPE AND CONSEQUENTLY EXTENSIVELY USED IN MISSILE EQUATIONS OF MOTION. THE EULER ANGLES USED IN THIS PAPER ARE TO BE UNDERSTOOD TO BE SUCCESSIVE TYPE EULER ANGLES UNLESS EXPLICITLY STATED OTHERWISE.

CONSIDER THE ORIENTATION OF A UNIT VECTOR $\overline{R}_1$ IN THE $\overline{F}_1$ REFERENCE FRAME. AS SHOWN IN SECTION ONE, THE THREE ANGLES THAT $\overline{R}_1$ MAKES WITH THE $\overline{F}_1$, $\overline{F}_2$ AND $\overline{F}_3$ AXES RESPECTIVELY ARE GIVEN BY α_1, β_1 AND γ_1. THE DIRECTION COSINE BETWEEN THE $\overline{R}_1$ AND $\overline{F}_1$ UNIT VECTOR IS $\cos \alpha_1 = \overline{R}_1 . \overline{F}_1$. IN TERMS OF EULER ANGLES (FOR EXAMPLE RADAR COORDINATES-AZIMUTH ANGLE ψ AND ELEVATION ANGLE Θ) AS SHOWN BELOW:

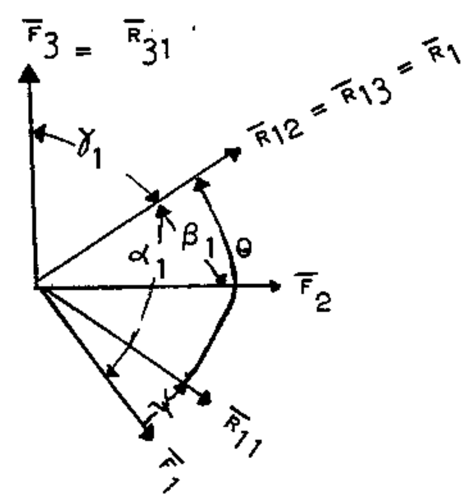


FIG. 2.1. DIRECTION ANGLES BETWEEN R_i AND F_i FRAMES.

THE RELATIONSHIP BETWEEN α_1, ψ AND Θ IS $\cos \alpha_1 = \cos \psi \cos \Theta$. IT IS TO BE NOTED THAT THIS IS $A_{11} = \overline{R}_1 . \overline{F}_1$ FOR THE ORDERED EULER SEQUENCE YAW, PITCH, ROLL, AND ALSO FOR THE SEQUENCE PITCH, YAW AND ROLL AS SHOWN IN FIGURES 2-1 AND 2-2.

THE TOTAL NUMBER OF ORDERED SETS OR SEQUENCES OF EULER ANGLES IS GIVEN BY THE PERMUTATIONS OF THREE ANGLES TAKEN THREE AT A TIME, OR SIX. THE ANGLES USED ARE ϕ, Θ, ψ, WHERE THE CORRESPONDENCE IS SUCH THAT ϕ, Θ, ψ ARE ALWAYS ASSOCIATED WITH ROTATIONS ABOUT THE $\overline{R}_{1J}$, $\overline{R}_{2J}$, $\overline{R}_{3J}$ (J = 1, 2, 3) VECTORS, respectively. The sequences of rotations are

$$\begin{array}{ccc}
\psi; & \theta, & \phi \\
\psi; & \phi; & \theta \\
\phi, & \psi, & \theta \\
\phi, & \theta, & \psi \\
\theta; & \phi; & \psi \\
\theta, & \psi, & \phi
\end{array}$$

The Euler Angles represent an ordered triple of numbers, thus for a unique orientation of one frame($\bar{R}_1$ with respect to a second frame $\bar{F}_1$) the angles ψ, θ, and ϕ for the sequence (ψ_1, θ_1, ϕ_1) will have different numerical values than the angles ψ_2, ϕ_2, and θ_2 for the sequence (ψ_2, ϕ_2, θ_2). This is shown in Figure 2-2 where the yaw angle and elevation angles for the first sequence represent respectively the normal projection of the $\bar{R}_1$ axis in the $\bar{F}_1$, $\bar{F}_2$ plane (heading angle for zero angle of attack) and the pitch angle is the elevation plane pitch angle. For the second sequence (ψ_2, ϕ_2, θ_2), ψ_2 is not heading angle and θ_2 is plane of symmetry pitch angle if a symmetrical airframe were assumed with $\bar{R}_1$ longitudinal axis and $\bar{R}_2$ left wing axis. Thus, unless an Euler sequence is specified, the three commonly denoted angles yaw, pitch, and roll are without specific meaning.

The transformation matrix M_Y associated with the angle will be referred to as the <u>yaw-matrix</u>. The M_Y matrix obtained is

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix}$$

$$\bar{R}_i = M_Y(\psi)\bar{F}_J \tag{2.1}$$

The matrix associated with the angle θ will be referred to as the <u>pitch matrix M_P</u>.

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix}$$

$$R_i = M_P(\theta)F_J \tag{2.2}$$

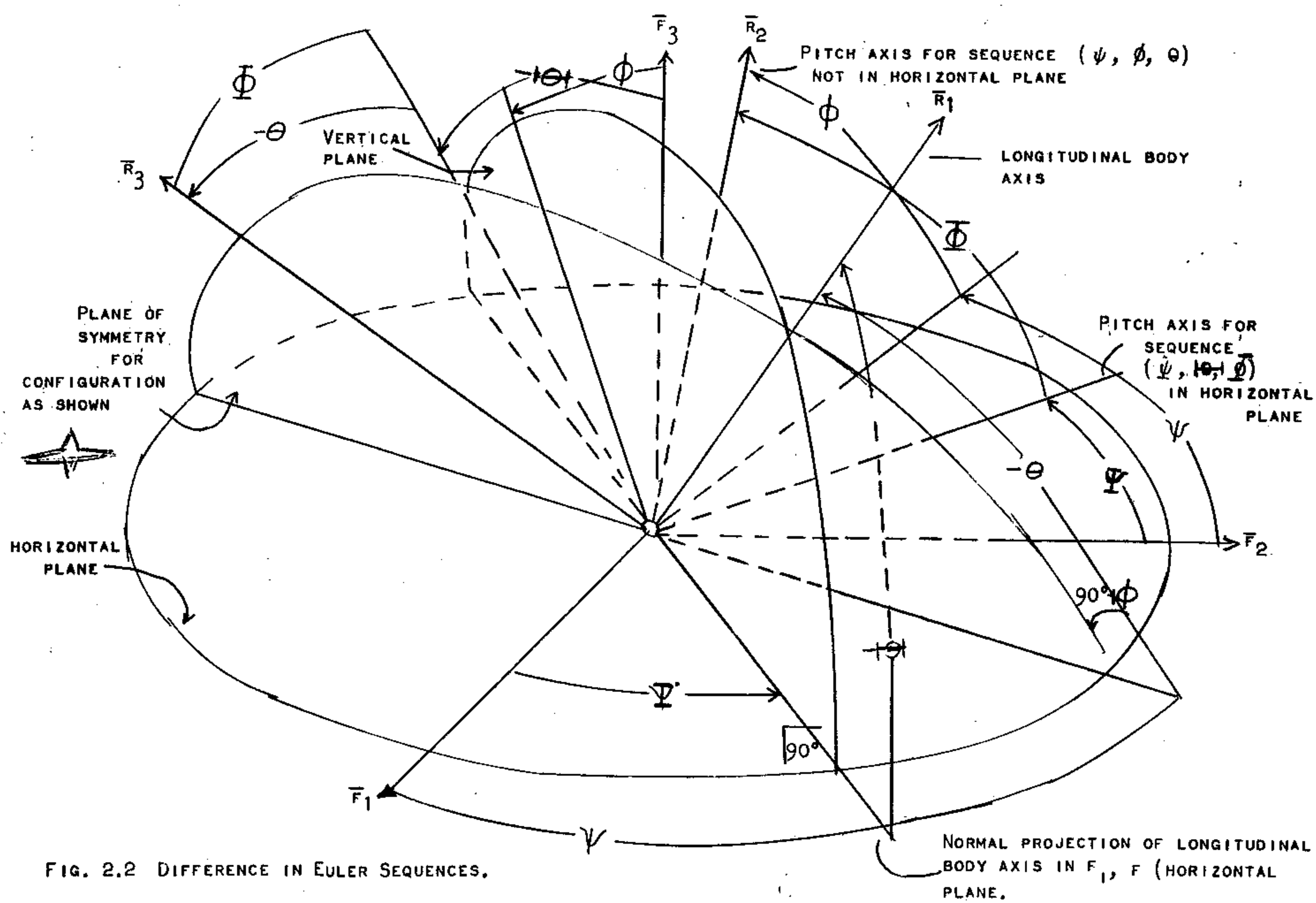


FIG. 2.2 DIFFERENCE IN EULER SEQUENCES.

Similarly, the matrix associated with the Euler angle ϕ will be referred to as the roll matrix M_R.

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} \tag{2.3}$$

$$\bar{R}_i = M_R(\phi)\bar{F}_J$$

The direction cosine matrix for the Euler sequence (ψ, Θ, ϕ) when the $\bar{R}_i$ frame is rotated with respect to the $\bar{F}_i$-frame will now be obtained.

The first angular rotation is about $\bar{R}_3 = \bar{F}_3$ by equation (2.1).

$$\bar{R}_{i1} = M_Y(\psi)\bar{F}_i$$

$$\begin{pmatrix} \bar{R}_{11} \\ \bar{R}_{21} \\ \bar{R}_{31} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} \tag{2.4}$$

The second angular rotation is a pitch about R_{21}. By Equation (2.2).

$$\bar{R}_{i2} = M_P(\Theta)\bar{R}_{J1}$$

$$\begin{pmatrix} \bar{R}_{12} \\ \bar{R}_{22} \\ \bar{R}_{32} \end{pmatrix} = \begin{pmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{pmatrix} \begin{pmatrix} \bar{R}_{11} \\ \bar{R}_{21} \\ \bar{R}_{31} \end{pmatrix} \tag{2.5}$$

The third angular rotation is a roll about $\bar{R}_1$, and is given by equation (2.3).

$$\bar{R}_i = M_R(\phi)\bar{R}_{i2}$$

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \bar{R}_{12} \\ \bar{R}_{22} \\ \bar{R}_{32} \end{pmatrix} \tag{2.6}$$

BY EQUATION (2.4), (2.5), AND (2.6)

$$\bar{R}_i = M_R M_P M_Y \bar{F}_i . \tag{2.7}$$

IN TERMS OF THE ANGLES, EQUATION (2.7) BECOMES

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} \tag{2.8}$$

MULTIPLYING THE ABOVE MATRICES GIVES THE MATRIX FOR THE SEQUENCE (ψ, θ, ϕ) AS

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\,\psi - \cos\phi\sin\psi) & (\sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi) & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) & (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} \tag{2.9}$$

THE SEQUENCE (ψ, ϕ, θ) IS GIVEN AS (SEE FIG. 2.2) $\bar{R}_{i1} = M_Y(\psi)\bar{F}_i$

$$\begin{pmatrix} \bar{R}_{11} \\ \bar{R}_{21} \\ \bar{R}_{31} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} \tag{2.10}$$

$$\bar{R}_{i2} = M_R(\phi)\bar{R}_{i1}$$

$$\begin{pmatrix} \bar{R}_{12} \\ \bar{R}_{21} \\ \bar{R}_{31} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \bar{R}_{11} \\ \bar{R}_{21} \\ \bar{R}_{31} \end{pmatrix} \tag{2.11}$$

OR

$$\overline{R}_i = M_P(\Theta)M_R(\phi)M_Y(\psi)\overline{F}_i$$

$$\begin{pmatrix} \overline{R}_1 \\ \overline{R}_2 \\ \overline{R}_3 \end{pmatrix} = \begin{pmatrix} (\cos\Theta\cos\psi - \sin\Theta\sin\phi\sin\psi) & (\cos\Theta\sin\psi + \sin\Theta\sin\phi\cos\psi) & -\sin\Theta\cos\phi \\ -\cos\phi\sin\psi & \cos\phi\cos\psi & \sin\phi \\ \sin\Theta\cos\psi + \cos\Theta\sin\phi\sin\psi) & (\sin\Theta\sin\psi - \cos\Theta\sin\phi\cos\psi) & \cos\Theta\cos\phi \end{pmatrix} \begin{pmatrix} \overline{F}_1 \\ \overline{F}_2 \\ \overline{F}_3 \end{pmatrix}$$

WHICH CORRESPONDS TO THE SEQUENCE ψ, ϕ, Θ.

IN A SIMILAR MANNER, ALL SIX TRANSFORMATION MATRICES ARE OBTAINED.

THE REMAINING FOUR SEQUENCES ARE OBTAINED FROM THE FOLLOWING MATRIX PRODUCTS.

$$(\phi, \psi, \Theta): \overline{R}_i = M_P(\Theta)\, M_Y(\psi)\, M_R(\phi)\, \overline{F}_i$$

$$(\phi, \Theta, \psi): \overline{R}_i = M_Y(\psi) M_P(\Theta)\, M_R(\phi)\, \overline{F}_i$$

$$(\Theta, \psi, \phi): \overline{R}_i = M_R(\phi) M_Y(\psi)\, M_P(\Theta)\, \overline{F}_i$$

$$(\Theta, \phi, \psi): \overline{R}_i = M_Y(\psi) M_R(\phi)\, M_P(\Theta)\, \overline{F}_i$$

THESE SIX SEQUENCES ARE DEPICTED IN FIGURES 3-1 THROUGH 3-6 WITH THE CORRESPONDING DIRECTION COSINE MATRICES. IT IS TO BE NOTED THAT IN ALL OF THE FOLLOWING FIGURES THE FRAME $\overline{R}_i$ IS ROTATED WITH RESPECT TO $\overline{F}_i$ THROUGH POSITIVE ANGLES. THUS, WHEN PARTICULAR VECTORS ARE SUBSTITUTED INTO THE EQUATIONS, THE FRAME WHICH IS BEING ORIENTED THROUGH THE EULER ANGLES WITH RESPECT TO A SECOND FRAME SHOULD BE PUT IN THE LEFT HAND SIDE OF THE EQUATION. IF NOT, ROWS AND COLUMNS OF THE DIRECTION COSINE MATRICES SHOULD BE INTERCHANGED, AS WELL AS THE SIGNS OF THE EULER ANGLES CHANGED, SINCE REVERSING THE TRANSFORMATIONS (GOING FROM THE $\overline{R}_i$ FRAME BACKWARDS TO THE $\overline{F}_i$ FRAME THROUGH THE SAME MAGNITUDES OF ANGLES) REQUIRES NEGATIVE ANGLES.

TRANSFORMATIONS WITH THE USE OF EULER ANGLES AND THEIR RATES

IN THE LAST SECTION, THE DIRECTION COSINES OF EQUATION 1-14 WERE OBTAINED AS TRIGONOMETRIC FUNCTIONS OF THE EULER ANGLES. WHEN THIS MEANS IS EMPLOYED TO OBTAIN THE DIRECTION COSINES, THE EULER ANGLES MUST BE GENERATED. IN THIS SECTION THE METHOD FOR GENERATING THE ANGLES IS GIVEN. ALSO IN THE NEXT SECTION, A SECOND METHOD FOR FINDING THESE DIRECTION COSINES DIRECTLY IS DESCRIBED. THIS LATTER METHOD REQUIRES THE SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS INVOLVING THE DIRECTION COSINES AND THEIR RESPECTIVE TIME DERIVATIVES

AND THE COMPONENTS OF ANGULAR VELOCITIES OF THE TWO REFERENCE FRAMES ABOUT THEIR RESPECTIVE AXES. ALTHOUGH THIS SYSTEM CONSISTS OF NINE FIRST ORDER DIFFERENTIAL EQUATIONS, ONLY THREE ARE INDEPENDENT.

THE DIFFERENTIAL EQUATIONS WHOSE SOLUTIONS GIVE THE EULER ANGLES FOR THE FORMER METHOD WILL BE OBTAINED FIRST. CONSIDER THE EULER SEQUENCE ψ, θ, ϕ WHICH ORIENTS THE $\overline{R}_i$ FRAME WITH RESPECT TO THE $\overline{F}_i$ FRAME. THE INERTIAL ANGULAR VELOCITY OF THE $\overline{R}_i$ FRAME IS EQUAL TO THE INERTIAL ANGULAR VELOCITY OF THE $\overline{F}_i$ FRAME PLUS THE RELATIVE ANGULAR VELOCITY OF THE $\overline{R}_i$ FRAME WITH RESPECT TO THE $\overline{F}_i$ FRAME, OR IN EQUATION FORM:

$$\overline{\omega}_r = \overline{\omega}_f + \overline{\omega}_{rf}. \tag{3.1}$$

IN COMPONENT FORM

$$P_R\overline{R}_1 + Q_R\overline{R}_2 + R_R\overline{R}_3 = P_F\overline{F}_1 + Q_F\overline{F}_2 + R_F\overline{F}_3 + \dot{\psi}\overline{F}_3 + \dot{\theta}\overline{R}_{21} + \dot{\phi}\overline{R}_1 \tag{3.2}$$

WHERE $\overline{R}_{21}$ IS FOUND FROM EQUATION (2.4) TO BE

$$\overline{R}_{21} = -\sin\psi\,\overline{F}_1 + \cos\psi\,\overline{F}_2 \tag{3.3}$$

USING (3.3) WITH (3.2)

$$\begin{aligned} P_R\overline{R}_1 + Q_R\overline{R}_2 + R_R\overline{R}_3 &= (P_F - \dot{\theta}\sin\psi)\overline{F}_1 \\ &+ (Q_F + \dot{\theta}\cos\psi)\overline{F}_2 \\ &+ (R_F + \dot{\psi})\overline{F}_3 + \dot{\phi}\overline{R}_1 \end{aligned} \tag{3.4}$$

DOTTING (3.4) WITH $\overline{R}_1$, $\overline{R}_2$, $\overline{R}_3$, RESPECTIVELY, UTILIZING EQUATION (2.9) AND SIMPLIFYING

$$P_R = P_F\cos\theta\cos\psi + Q_F\cos\theta\sin\psi - R_F\sin\theta + \dot{\phi} - \dot{\psi}\sin\theta. \checkmark \tag{3.5}$$

$$\begin{aligned} Q_R = {} & P_F(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + Q_F(\sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi) \\ & + R_F\sin\phi\cos\theta + \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta. \end{aligned} \tag{3.6}$$

$$\begin{aligned} R_F = {} & P_F(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) + P_F(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \\ & + R_F\cos\phi\sin O - \theta\sin\phi + \psi\cos\theta\cos\phi. \end{aligned} \tag{3.7}$$

EQUATIONS (3.5), (3.6), AND (3.7) ARE THE DIFFERENTIAL EQUATIONS WHOSE SOLUTIONS GIVE THE EULER ANGLES OF THE SEQUENCE ψ, θ, ϕ WHEN THE SIX COMPONENTS ω_{Ri}, ω_{Fi} ARE KNOWN AND ARE USED WITH EQUATIONS (2.9) TO FIND THE DIRECTION COSINES.

IN CASE $\overline{\omega}_F = 0$, THAT IS $\overline{F}_i$ IS AN INERTIAL FRAME, EQUATIONS (3.5), (3.6), AND (3.7) ARE

$$P_R = \dot{\phi} - \dot{\psi}\sin\theta \tag{3.8}$$

$$Q_R = \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \tag{3.9}$$

$$R_R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \tag{3.10}$$

WHEN THESE EQUATIONS ARE SOLVED FOR $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$, ONE OBTAINS

$$\dot{\psi} = (Q_R\sin\phi + R_R\cos\phi)\sec\theta \tag{3.11}$$

$$\dot{\theta} = Q_R\cos\phi - R_R\sin\phi \tag{3.12}$$

$$\dot{\phi} = P_R + (Q_R\sin\phi + R_R\cos\phi)\tan\theta \tag{3.13}$$

THE EQUATIONS FOR THE SEQUENCE ψ, ϕ, θ ARE OBTAINED WHEN $\overline{\omega}_{RF}$ OF EQUATIONS (3.1) IS SET EQUAL TO

$$\overline{\omega}_{R/F} = \dot{\psi}\overline{F}_3 + \dot{\phi}\overline{R}_{11} + \dot{\theta}\overline{R}_2 \tag{3.14}$$

WHERE $\overline{R}_{11}$ IS OBTAINED USING EQUATION (2.10) AS

$$\overline{R}_{11} = \overline{F}_1\cos\psi + \overline{F}_2\sin\psi \tag{3.15}$$

UTILIZING EQUATION (3.14) AND (3.15) IN (3.1)

$$\begin{aligned} P_R\overline{R}_1 + Q_R\overline{R}_2 + R_R\overline{R}_3 &= (P_F + \dot{\phi}\cos\psi)\overline{F}_1 \\ &+ (Q_F + \dot{\phi}\sin\psi)\overline{F}_2 + (R_F + \dot{\psi})\overline{F}_3 + \dot{\theta}\overline{R}_2. \end{aligned} \tag{3.16}$$

DOTTING EQUATION (3.16) WITH $\overline{R}_1$, $\overline{R}_2$, $\overline{R}_3$ RESPECTIVELY, UTILIZING THE DIRECTION COSINE MATRIX OF EQUATION (2.13) AND SIMPLIFYING

$$\begin{aligned} P_R &= P_F(-\sin\theta\sin\phi\sin\psi + \cos\theta\cos\psi) \\ &+ Q_F(\sin\theta\sin\phi\cos\psi + \cos\theta\sin\psi) \\ &- R_F\sin\theta\cos\phi + \dot{\phi}\cos\theta - \dot{\psi}\sin\theta\cos\phi \end{aligned} \tag{3.17}$$

$$Q_R = -P_F\cos\phi\sin\psi + Q_F\cos\phi\cos\psi + R_F\sin\phi$$
$$+ \dot{\theta} + \dot{\psi}\sin\phi \tag{3.18}$$

$$R_R = P_F(\cos\theta\sin\phi\sin\psi + \sin\theta\cos\psi) + Q_F(-\cos\theta\sin\phi\cos\psi$$
$$+ \sin\theta\sin\psi) + R_F\cos\theta\cos\phi + \dot{\phi}\sin\theta + \dot{\psi}\cos\theta\cos\phi \tag{3.19}$$

EQUATIONS (3.17), (3.18) AND (3.19) ARE THE DIFFERENTIAL EQUATIONS WHOSE SOLUTIONS GIVE THE EULER ANGLES OF THE SEQUENCE ψ, ϕ, θ WHEN THE SIX COMPONENTS $\overline{\omega}_{RI}$, $\overline{\omega}_{FI}$ ARE KNOWN. THE SOLUTIONS USED WITH EQUATION (2.13) YIELD THE DIRECTION COSINES. FURTHERMORE, IF THE EULER ANGULAR RATES ($\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$) AND $\overline{\omega}_F$ (OR $\overline{\omega}_R$) ARE KNOWN THEN $\overline{\omega}_R$ (OR $\overline{\omega}_F$) CAN BE DETERMINED USING EQUATIONS (3.17), (3.18) AND (3.19).

IF $\overline{\omega}_F = \overline{0}$ THEN EQUATIONS (3.17), (3.18) AND (3.19) REDUCE TO

$$P_R = \dot{\phi}\cos\theta - \dot{\psi}\sin\theta\cos\phi \tag{3.20}$$

$$Q_R = \dot{\theta} + \dot{\psi}\sin\phi \tag{3.21}$$

$$R_R = \dot{\phi}\sin\theta + \dot{\psi}\cos\theta\cos\phi \tag{3.22}$$

SOLVING THESE EQUATIONS FOR

$$\dot{\psi} = (R_R\cos\theta - P_R\sin\theta)\sec\phi \tag{3.23}$$

$$\dot{\phi} \quad P_R\cos\theta + R_R\sin\theta \tag{3.24}$$

$$\dot{\theta} = Q_R + (P_R\sin\theta - R_R\cos\theta)\tan\phi \tag{3.25}$$

IF THE COMPONENTS OF $\overline{\omega}_{RI}$ IN THE LATTER DIFFERENTIAL EQUATIONS ARE KNOWN, THE SOLUTIONS YIELD THE DESIRED EULER ANGLES FROM WHICH THE DIRECTION COSINES OF FIGURE 3-1 CAN BE OBTAINED. IT IS WORTHWHILE TO NOTE THAT $\dot{\psi}$ AND $\dot{\theta}$ ARE NOT DEFINED WHEN $\phi = \pm 90°$.

THE THREE DIFFERENTIAL EQUATIONS FOR EACH OF THE REMAINING EULER SEQUENCES MAY BE OBTAINED IN A SIMILAR MANNER.

SUMMARIZING THE SIX VECTOR EQUATIONS FOR THE SIX EULER SEQUENCES, ONE OBTAINS

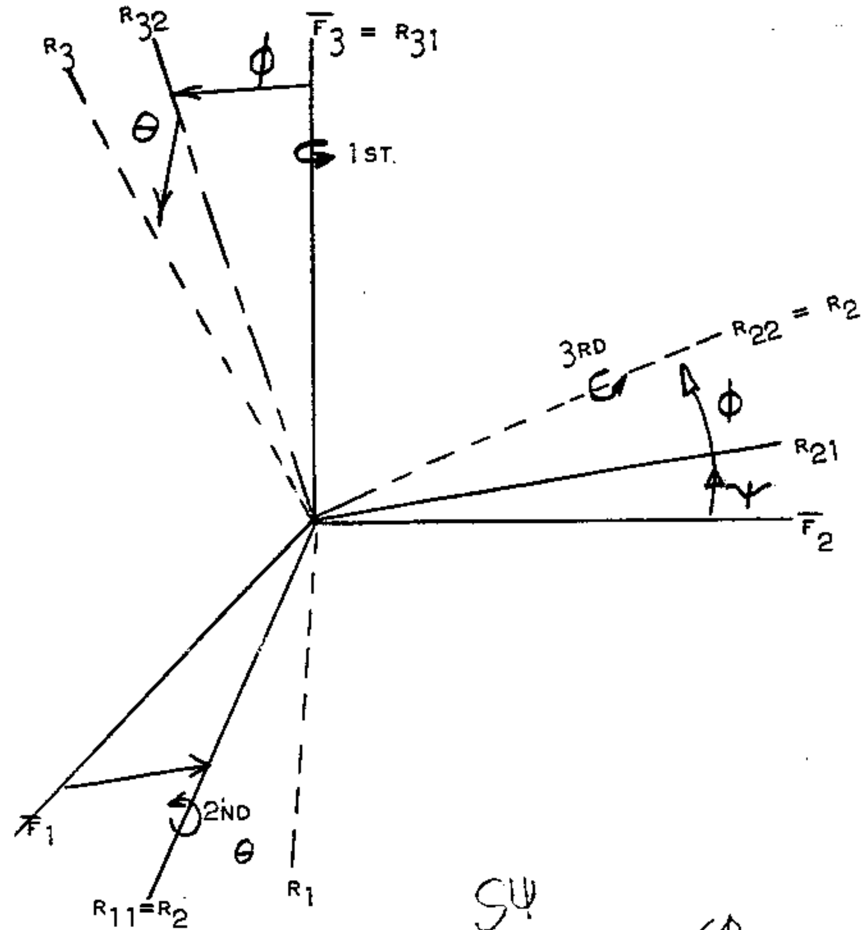
$$(\psi, \theta, \phi): \overline{\omega}_R = \overline{\omega}_F + \dot{\psi}\overline{F}_3 + \dot{\theta}\overline{R}_{21} + \dot{\phi}\overline{R}_1$$
$$(\psi, \phi, \theta): \overline{\omega}_R = \overline{\omega}_F + \dot{\psi}\overline{F}_3 + \dot{\phi}_1\overline{R}_{11} + \dot{\theta}\overline{R}_2$$
$$(\phi, \theta, \psi): \overline{\omega}_R = \overline{\omega}_F + \dot{\phi}\overline{F}_1 + \dot{\theta}\overline{R}_{21} + \dot{\psi}\overline{R}_3 \tag{3.26}$$

$$(\phi, \psi, \theta): \bar{\omega}_R = \bar{\omega}_F + \dot{\phi}\bar{F}_1 + \dot{\psi}\bar{R}_{31} + \dot{\theta}\bar{R}_2$$

$$(\theta, \phi, \psi)\ \bar{\omega}_R \quad \bar{\omega}_F \quad \dot{\theta}\bar{F}_2 + \dot{\phi}\bar{R}_{11} + \dot{\psi}\bar{R}_3$$

$$(\theta, \psi, \phi): \bar{\omega}_R = \bar{\omega}_F + \dot{\theta}\bar{F}_2 \quad \dot{\psi}\bar{R}_{31} \quad \dot{\phi}\bar{R}_1 \tag{3.26}$$

THE THREE SCALAR EQUATIONS OBTAINED FOR EACH OF THE EULER SEQUENCES OF (3-26) (A TOTAL OF 18 SCALAR EQUATIONS) ARE GIVEN IN FIGURES 3-1 TO 3-6 FOR THE RESPECTIVE SEQUENCE.



$S\psi$ $S\phi$

	$\bar{F}_1$	$\bar{F}_2$	$\bar{F}_3$
$\bar{R}_1$	$-s\theta s\phi s\psi + c\theta c\psi$	$s\theta s\phi c\psi + c\theta s\psi$	$-s\theta c\phi$
$\bar{R}_2$	$-c\phi s\psi$	$c\phi c\psi$	$s\phi$
$\bar{R}_3$	$c\theta s\phi s\psi + s\theta c\psi$	$-c\theta s\phi c\psi + s\theta s\psi$	$c\theta c\phi$

$$P_R = \dot{\phi}\cos\theta - \dot{\psi}\sin\theta\cos\phi + K_1 \qquad K_i = P_F A_{i1} + Q_F A_{i2} + R_F A_{i3}$$

$$Q_R = \dot{\theta} + \dot{\psi}\sin\phi + K_2 \qquad i = 1,\ 2,\ 3$$

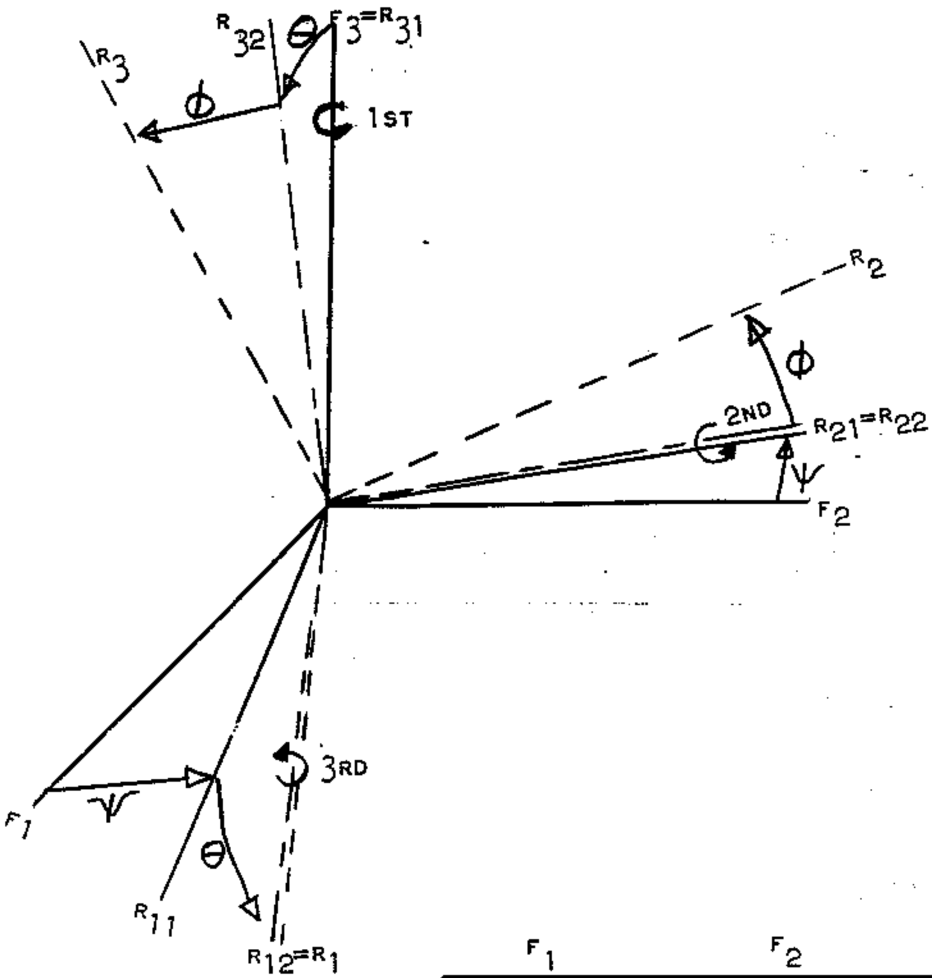
$$R_R = \dot{\phi}\sin\theta + \dot{\psi}\cos\theta\cos\phi + K_3$$

$$\dot{\phi} = \cos\theta(P_R - K_1) + \sin\theta(R_R - K_3)$$

$$\dot{\theta} = -\tan\phi\cos\theta(R_R - K_3) + (Q_R - K_2) + \tan\phi\sin\theta(P_R - K_1)$$

$$\dot{\psi} = \left[\cos\theta(R_R - K_3) - \sin\theta(P_R - K_1)\right]\sec\phi$$

FIG. 3-1. SEQUENCE ψ, ϕ, θ.



	F_1	F_2	F_3
R_1	$c\theta c\psi$	$c\theta s\psi$	$-s\theta$
R_2	$s\phi s\theta c\psi - c\phi s\psi$	$s\phi s\theta s\psi + c\phi c\psi$	$s\phi c\theta$
R_3	$c\phi s\theta c\psi + s\phi s\psi$	$c\phi s\theta s\psi - s\phi c\psi$	$c\phi c\theta$

$P_R = \dot{\phi} - \dot{\psi}\sin\theta + K_1$ $\qquad K_1 = P_F A_{i1} + Q_F A_{i2} + R_F A_{i3}$

$Q_R = \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta + K_2$ $\qquad i = 1, 2, 3.$

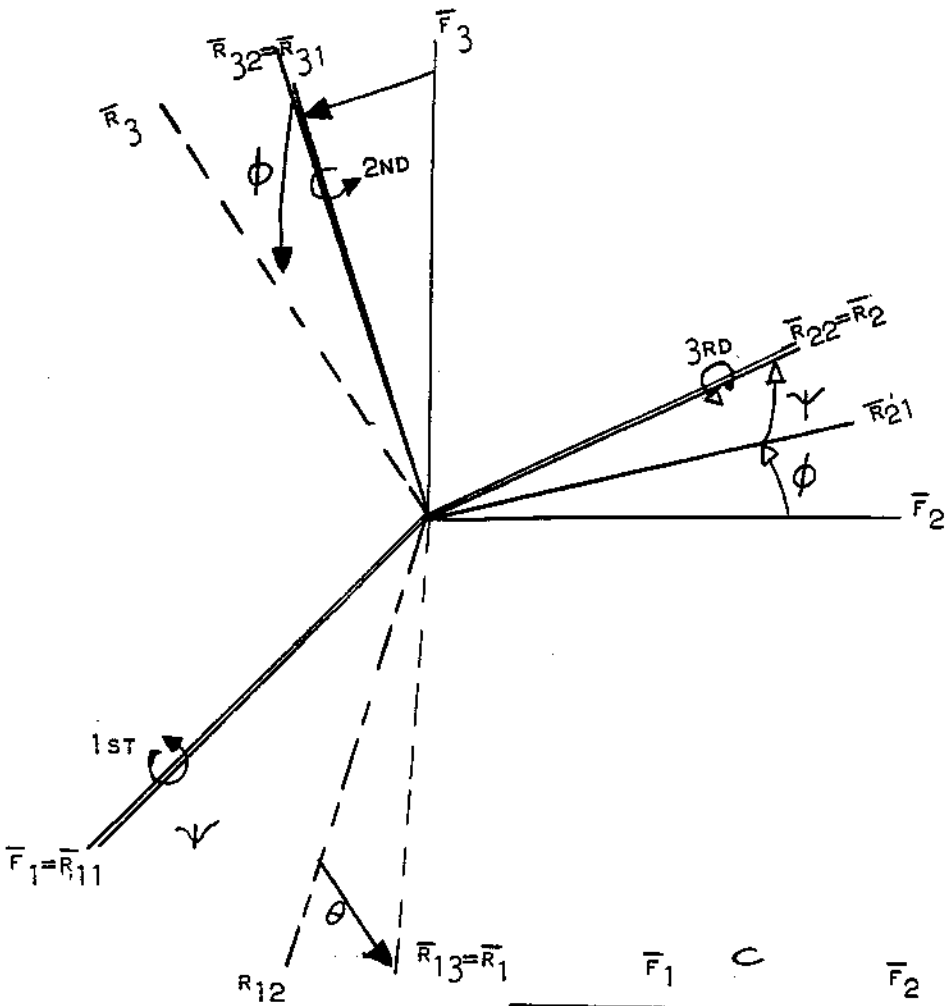
$R_R = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi + K_3$

$$\dot{\phi} = (P_R - K_1) + \tan\bar{\theta}\cos\phi\,(R_R - K_3) + \tan\theta\sin\phi(Q_R - K_2)$$

$$\dot{\psi} = [\sin\phi(Q_R - K_2) + \cos\phi(R_R - K_3)]\sec\theta$$

$$\dot{\theta} = +\cos\phi(Q_R - K_2) - \sin\phi(R_R - K_3)$$

FIG. 3-2. SEQUENC ψ, θ, ϕ.



	$\overline{F}_1$	$\overline{F}_2$	$\overline{F}_3$
$\overline{R}_1$	$c\theta c\psi$	$c\theta s\psi c\phi + s\theta s\phi$	$s\psi s\phi c\theta - s\theta c\phi$
$\overline{R}_2$	$-s\psi$	$c\phi c\psi$	$c\psi s\phi$
$\overline{R}_3$	$s\theta c\psi$	$s\theta s\psi c\phi - c\theta s\phi$	$s\theta s\phi s\psi + c\theta c\phi$

$$P_R = \dot{\phi}\cos\theta\cos\psi - \dot{\psi}\sin\theta + K_1$$

$$Q_R = \dot{\theta} - \dot{\phi}\sin\psi + K_2$$

$$R_R = \dot{\phi}\sin\theta\cos\psi + \dot{\psi}\cos\theta + K_3$$

$$\dot{\phi} = [\sin\theta(R_R - K_3) + \cos\theta(P_R - K_1)]\sec\psi$$

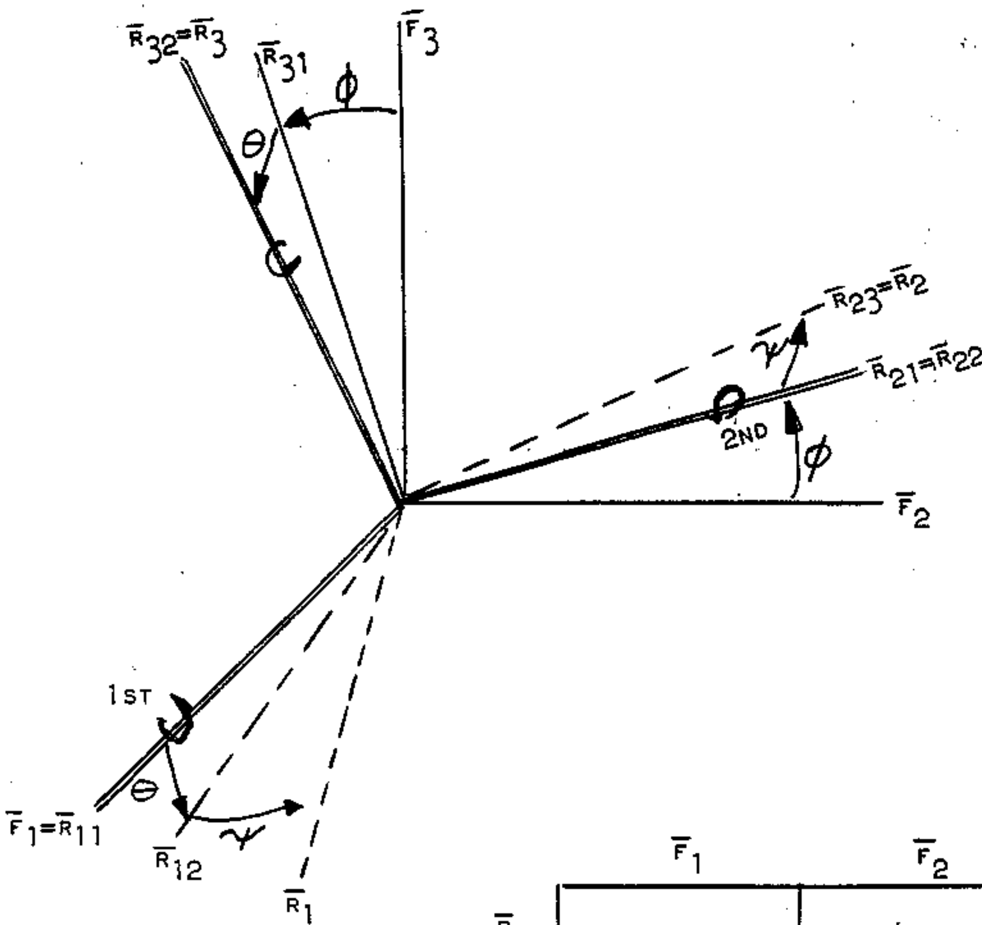
$$\dot{\psi} = \cos\theta(R_R - K_3) - \sin\theta(P_R - K_1)$$

$$\dot{\theta} = (Q_R - K_2) + \tan\psi\cos\theta(P_R - K_1) + \tan\psi\sin\theta(R_R - K_3)$$

$$K_i = P_F A_{i1} + Q_F A_{i2} + R_F A_{i3}$$

$$i = 1, 2, 3$$

FIG. 3-3. SEQUENCE ϕ, ψ, θ.



	$\bar{F}_1$	$\bar{F}_2$	$\bar{F}_3$
$\bar{R}_1$	$c\theta c\psi$	$c\psi\,\theta s\phi + s\psi\,c\phi$	$-s\theta c\phi c\psi + s\psi s\phi$
$\bar{R}_2$	$-s\psi\,c\theta$	$-s\psi\,s\theta s\phi + c\psi\,c\phi$	$s\theta s\psi\,c\phi + c\psi\,s\phi$
$\bar{R}_3$	$s\theta$	$-s\phi c\theta$	$c\theta c\phi$

$$P_R = \dot{\phi}\cos\theta\cos\psi + \dot{\theta}\sin\psi + K_1$$

$$Q_R = \dot{\theta}\cos\psi - \dot{\phi}\sin\psi\cos\theta + K_2$$

$$R_R = \dot{\phi}\sin\theta + \dot{\psi} + K_3$$

$$\dot{\psi} = [\cos\psi\, -P_R-K_1) + \sin\psi(Q_R-K_2)]\sec\theta$$

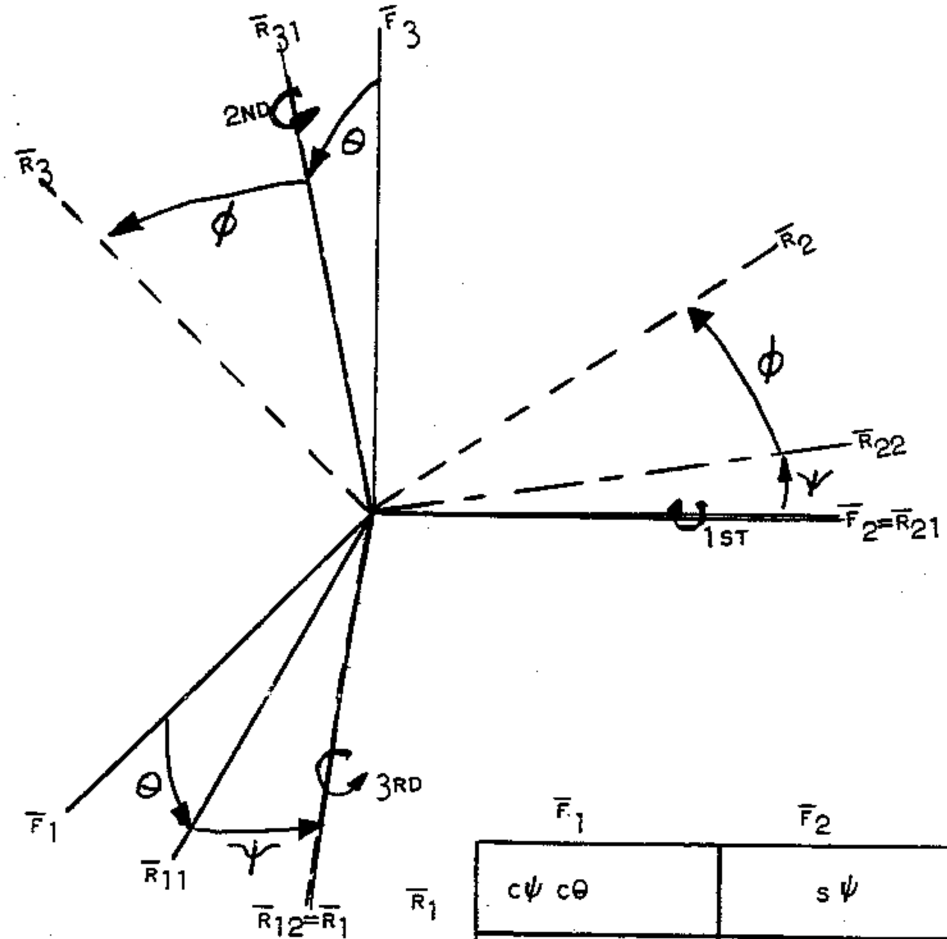
$$\dot{\theta} = \cos\psi(Q_R-K_2) + \sin\psi\, P_R-K_1)$$

$$\dot{\phi} = (R_R-K_3) + \sin\psi\tan\theta(Q_R-K_2) - \cos\psi\tan\theta(P_R-K_1)$$

$$K_i = P_F A_{iL} + Q_F A_{i2} + R_F A_{i3}$$

$i = 1, 2, 3.$

FIG. 3-4. SEQUENCE ϕ, θ, ψ .



	$\overline{F}_1$	$\overline{F}_2$	$\overline{F}_3$
$\overline{R}_1$	$c\psi\ c\theta$	$s\psi$	$-c\psi\ s\theta$
$\overline{R}_2$	$-c\phi s\psi\ c\theta + s\phi s\theta$	$c\phi c\psi$	$c\phi s\psi\ s\theta + s\phi c\theta$
$\overline{R}_3$	$s\phi s\psi\ c\theta + c\phi s\theta$	$-s\phi c\psi$	$-s\phi s\psi s\theta + c\phi c\theta$

$$P_R = \dot{\phi} + \dot{\theta}\sin\psi + K_1 \qquad K_I = P_F A_{12} + Q_F A_{12} + R_F A_{13}$$

$$Q_R = \dot{\psi}\sin\phi + \dot{\theta}\cos\phi\cos\psi + K_2 \qquad I = 1,\ 2,\ s$$

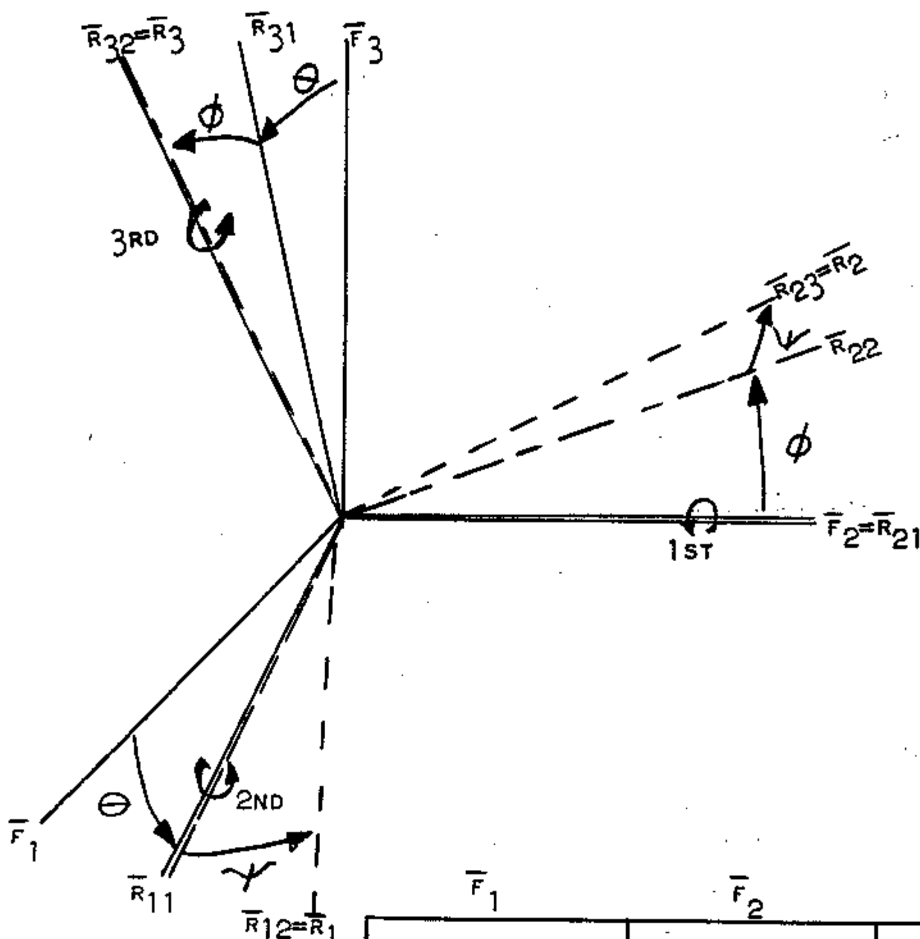
$$R_R = \dot{\psi}\cos\phi - \dot{\theta}\sin\phi\cos\psi + K_3$$

$$\dot{\psi} = \cos\phi(R_R - K_3) + \sin\phi(Q_R - K_2)$$

$$\dot{\phi} = (P_R - K_1) + \tan\psi\sin\phi(R_R - K_3) - \tan\psi\cos\phi(Q_R - K_2)$$

$$\dot{\theta} = [\cos\phi(Q_R - K_2) - \sin\phi(R_R - K_3)]\sec\psi$$

FIG. 3-5. SEQUENCE θ, ψ, ϕ



	$\bar{F}_1$	$\bar{F}_2$	$\bar{F}_3$
$\bar{R}_1$	$s\psi s\phi s\theta + c\psi c\theta$	$s\psi c\phi$	$s\psi s\phi c\theta - c\psi s\theta$
$\bar{R}_2$	$c\psi s\phi s\theta - s\psi c\theta$	$c\psi c\phi$	$c\psi s\phi c\theta + s\psi s\theta$
$\bar{R}_3$	$c\phi s\theta$	$-s\phi$	$c\phi c\theta$

$$P_R = \dot{\phi}\cos\psi + \dot{\theta}\sin\psi\cos\phi + K_1 \qquad K_i = P_F A_{i2} + Q_F A_{i2} + R_F A_{i3}$$

$$Q_R = \dot{\theta}\cos\phi\cos\psi - \dot{\phi}\sin\psi + K_2 \qquad i = 1,\ 2,\ 3$$

$$R_R = \dot{\psi} - \dot{\theta}\sin\phi + K_3$$

$$\dot{\phi} = \cos\psi\,(P_R-K_1) - \sin\ (Q_R-K_2)$$

$$\dot{\theta} = \left[\cos\psi\,(Q_R-K_2) + \sin\psi\,(P_R-K_1)\right]\sec\phi$$

$$\dot{\psi} = +\tan\phi\,\sin\psi\,(P_R-K_1) + (R_R-K_3) + \tan\phi\cos\psi\,(Q_R-K_2)$$

FIG. 3-6. SEQUENCE θ, ϕ, ψ

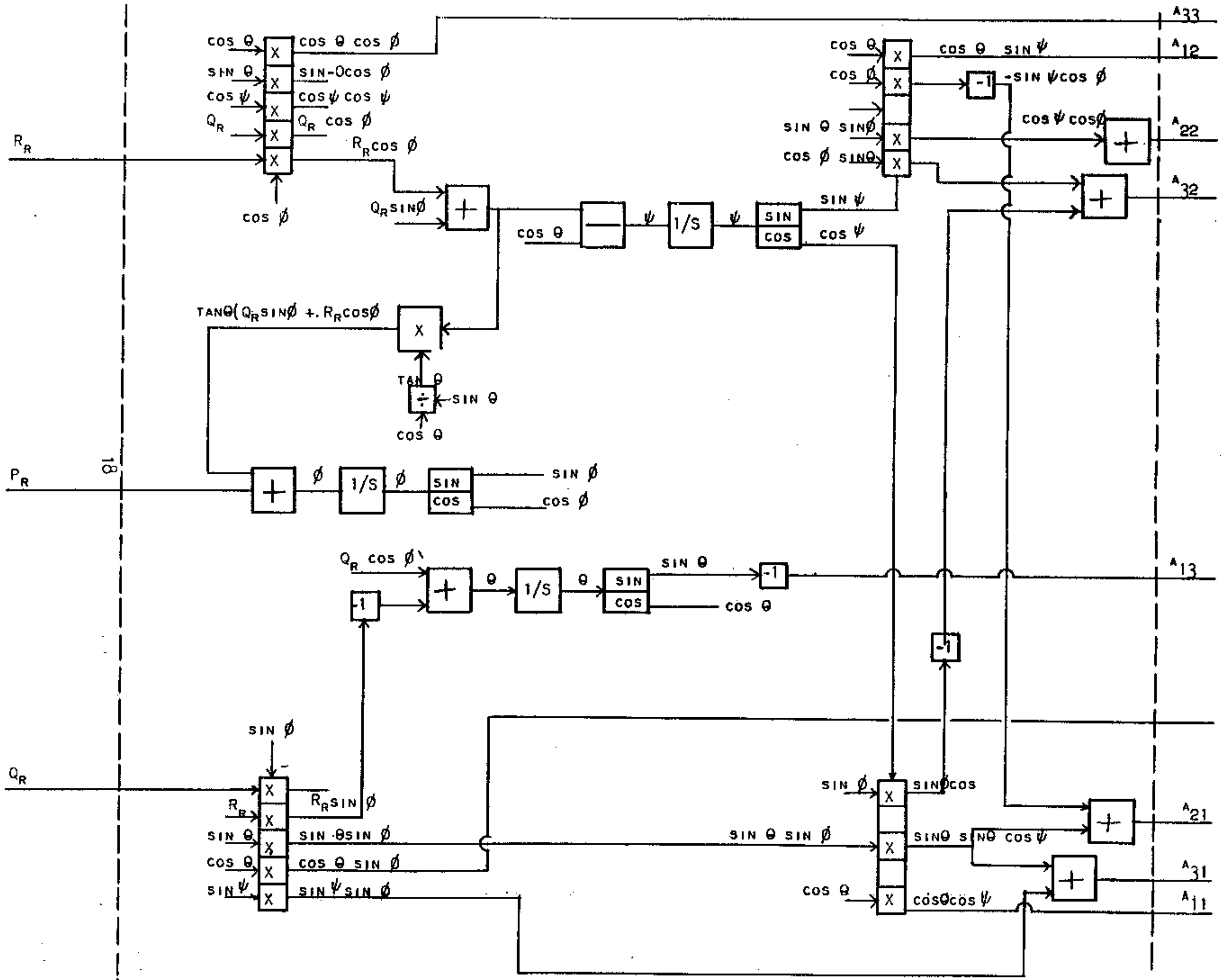


FIG. 4. MECHANIZATION FOR DIRECTION COSINES OF SEQUENCE ψ, θ, ϕ.

PROBLEM EXAMPLE

GIVEN THE ORIENTATION OF THE MISSILE BODY AXES ($\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$) WITH RESPECT TO A GROUND REFERENCE FRAME ($\bar{G}_1$, $\bar{G}_2$, $\bar{G}_3$) THROUGH THE FOLLOWING ORIENTATION YAW, PITCH, ROLL, (ψ, θ, ϕ), WHERE

$$\Psi = 30^\circ$$

$$\Theta = 45^\circ$$

$$\Phi = 0$$

TO FIND THE VALUES OF THE THREE ANGLES OF THE SEQUENCE PITCH, YAW, ROLL (θ, ψ, ϕ) ORIENTING THE SAME BODY AXES WITH RESPECT TO THE GROUND REFERENCE FRAME.

THE DIRECTION COSINE MATRICES FOR THE SEQUENCE (Ψ, Θ, Φ) AND (θ, ψ, ϕ) ARE GIVEN IN FIGURES 3-2 AND 3-5 RESPECTIVELY AS

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} c\Theta c\Psi & c\Theta s\Psi & -s\Theta \\ s\Phi s\Theta c\Psi - c\Phi s\Psi & s\Phi s\Theta s\Psi + c\Phi c\Psi & s\Phi c\Theta \\ c\Phi s\Theta c\Psi + s\Phi s\Psi & c\Phi s\Theta s\Psi - s\Phi c\Psi & c\Phi c\Theta \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} B^*_{11} & B^*_{12} & B^*_{13} \\ B^*_{21} & B^*_{22} & B^*_{23} \\ B^*_{31} & B^*_{32} & B^*_{33} \end{pmatrix}}_{(\Psi, \Theta, \Phi)} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} \qquad (3\text{-}27)$$

AND

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} c\psi c\theta & s\psi & -c\psi s\theta \\ -c\phi s\psi c\theta + s\phi s\theta & c\phi c\psi & c\phi s\psi s\theta + s\phi c\theta \\ s\phi s\ c\theta + c\phi s\theta & -s\phi c\ -s\phi c & -s\phi s\psi s\theta + c\phi c\theta \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}}_{(\theta, \psi, \phi)} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} \qquad (3\text{-}28)$$

THE BODY AXES (B_I) OF EQUATION (3-27) EQUALS (B_I) OF EQUATION (3-28) AND (G_J) OF EQUATION (3-27) EQUALS (G_J) OF EQUATION (3-28), THEREFORE THE TWO DIRECTION COSINE MATRICES ARE EQUAL, AND HENCE CORRESPONDING ELEMENTS OF THE DIRECTION COSINE MATRICES ARE EQUAL.

EQUATING CORRESPONDING ELEMENTS OF THE FIRST ROW, SECOND COLUMN $B^*_{12} = B_{12}$

$$s\psi = c\Theta\, s\Psi \tag{3-29}$$

OR

$$\psi = \sin^{-1}(c\Theta\, s\Psi). \tag{3-30}$$

EQUATING RATIOS OF THE ELEMENTS OF THE FIRST ROW, THIRD COLUMN TO THE FIRST ROW FIRST COLUMN OF CORRESPONDING MATRICES

$$\frac{B^*_{13}}{B^*_{11}} = \frac{B_{13}}{B_{11}} = \frac{-s\Theta}{c\,\Theta\, c\psi} = \frac{-c\psi s\theta}{c\psi\, c\theta},$$

$$-\tan\theta = -\tan\Theta\sec\psi. \tag{3-31}$$

TO SOLVE FOR ϕ, SET

$$\frac{B_{32}}{B_{22}} = \frac{B^*_{32}}{B^*_{22}},$$

OR

$$\frac{-s\phi c\psi}{c\phi\; c\psi} = \frac{c\Phi s\,\Theta\, s\Psi - s\,\Phi c\Psi}{s\Phi s\,\Theta\, s\Psi + c\,\Phi c\Psi} \tag{3-32}$$

DIVIDING NUMERATOR AND DENOMINATOR OF EQUATION (3-32) BY A $c\,\Phi\, c\Psi$ ONE OBTAINS

$$-\tan\phi = \frac{s\,\Theta\tan\Psi - \tan\Phi}{\tan\Phi(s\,\Theta\tan\Psi) + 1} \tag{3-33}$$

SINCE $\tan(x-y) = \dfrac{\tan x - \tan y}{\tan x\tan y + 1}$

LET $s\Theta\tan\Psi = \tan x$

$\tan\Phi = \tan y$,

THEN $x = \tan^{-1}(s\Theta\tan\Psi)$

$y = \Phi$,

AND

$$-\tan\phi = \tan\left[\tan^{-1}(s\Theta\tan\Psi) - \Phi\right],$$

OR

$$\phi = \Phi - \tan^{-1}\left[s\Theta\tan\Psi\right] \tag{3-34}$$

Using $\Psi = 30°$, $\Theta = 45°$, $\Phi = 0°$ and equations (3-30), (3-31) and (3-34)

$$\theta = 49°6'$$

$$\psi = 20°42'$$

$$\phi = 21°54' \qquad (3\text{-}35)$$

To verify the results, substituting the triple (30°, 45°, 0°) into the direction cosine matrix of equation (3-27) for (Ψ, Θ, Φ) yields approximately

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} .61237 & .35356 & -.70711 \\ -.50000 & .86603 & 0 \\ .61237 & .35356 & .70711 \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix}_{(\Psi, \Theta, \Phi)}$$

Substituting the results of equation (3-35) into the sequence (θ, ψ, ϕ) of equation (3-28) yields

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} .6124 & .3535 & -.70677 \\ -.4966 & .86790 & .00370 \\ .61478 & .34889 & .70715 \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix}_{(\theta, \psi, \phi)}$$

SEQUENCE CONVERSION

It appears that the Euler sequence yaw, pitch, roll (Ψ, Θ, Φ) orienting the missile body axes with respect to the ground axes are of the nature of a "preferred" sequence. This sequence of angles is quite often either directly or indirectly used to "control" the attitude of the missile. This seems quite natural when one considers that the yaw angle Ψ of this sequence is the ground heading angle when the missile has no angle of attack, that the pitch angle Θ (elevation plane pitch angle) is the missile climb or dive angle in the vertical plane.

As a result of the "preferred" nature of the sequence $(\bar{\Psi}, \bar{\Theta}, \bar{\phi})$ the conversion between the angles of this sequence and the other five sequences have been made in the following equations. In all cases these relations are obtained from equating the matrices of Figures 3-1 to 3-6.

The matrix for the sequence $(\bar{\Psi}, \bar{\Theta}, \bar{\phi})$ is denoted by $M(\bar{\Psi}, \bar{\Theta}, \bar{\phi})$ and similarly for the other sequences.

<u>Conversion Between $(\bar{\Psi}, \bar{\Theta}, \bar{\phi})$ and (ψ, ϕ, θ)</u>

Equating the two matrices of Figures 3-2 and 3-1

$$M(\bar{\Psi}, \bar{\Theta}, \bar{\phi}) = M(\psi, \phi, \theta),$$

To find $\bar{\Theta}$:

$$B^*_{13} = B_{13}$$

$$\sin \bar{\Theta} = \sin \theta \cos \phi$$

$$\bar{\Theta} = \sin^{-1}(\sin\theta\cos\phi)$$

To find $\bar{\phi}$:

$$\frac{B^*_{23}}{B^*_{33}} = \frac{B_{23}}{B_{33}}$$

$$\frac{\sin \bar{\phi} \cos \bar{\Theta}}{\cos \bar{\phi} \cos \bar{\Theta}} = \frac{\sin \phi}{\cos \theta \cos \phi}$$

$$\tan \bar{\phi} = \sec \theta \tan \phi$$

$$\bar{\phi} = \tan^{-1}(\sec \theta \tan \phi)$$

To find $\bar{\Psi}$:

$$\frac{B^*_{12}}{B^*_{11}} = \frac{B_{12}}{B_{11}}$$

$$\frac{\cos \bar{\Theta} \sin \bar{\Psi}}{\cos \bar{\Theta} \cos \bar{\Psi}} = \frac{\sin \theta \sin \phi \cos \psi + \cos \theta \sin \psi}{-\sin \theta \sin \phi \sin \psi + \cos \theta \cos \psi}$$

$$\tan\psi = \frac{\dfrac{\sin\theta\sin\phi\cos\psi}{\cos\theta\cos\psi} + \dfrac{\cos\theta\sin\psi}{\cos\theta\cos\psi}}{-\dfrac{\sin\text{-}0\sin\phi\sin\psi}{\cos\theta\cos\psi} + \dfrac{\cos\theta\cos\psi}{\cos\theta\cos\psi}}$$

$$\tan\psi = \frac{\tan\text{-}0\sin\phi + \tan\psi}{1 - \tan\theta\sin\phi\tan\psi}$$

LET $\tan\theta\sin\phi = \tan X$; LET $\tan\psi = \tan Y$

$$\tan\psi = \frac{\tan X + \tan Y}{1 - \tan X\tan Y} = \tan(X + Y)$$

$$X = \tan^{-1}(\tan\theta\sin\phi)$$

$$Y = \psi$$

$$\tan\psi = \tan\left[\tan^{-1}(\tan\theta\sin\phi) + \psi\right]$$

$$= \tan^{-1}(\tan\theta\sin\phi) + \psi$$

TO FIND ϕ, LET: $B_{23} = B^*_{23}$

$$\sin\phi = \sin\bar{\phi}\cos\cancel{\theta}$$

$$\phi = \sin^{-1}(\sin\bar{\phi}\cos\cancel{\theta})$$

TO FIND θ, LET: $\dfrac{B_{13}}{B_{33}} = \dfrac{B^*_{13}}{B^*_{33}}$

$$\frac{\sin\theta\cos\phi}{\cos\theta\cos\phi} = \frac{\sin\cancel{\theta}}{\cos\bar{\phi}}\cos 0$$

$$\tan\theta = \sec\phi\tan\bar{\theta}$$

$$\theta = \tan^{-1}(\sec\phi\tan\theta)$$

TO FIND ψ LET:

$$\frac{B_{21}}{B_{22}} = \frac{B^*_{21}}{B^*_{22}}$$

$$\frac{-\cos\phi\,\sin\psi}{\cos\phi\,\cos\psi} = \frac{\sin\bar{\phi}\,\sin\bar{\theta}\,\cos\bar{\psi}\;-\cos\bar{\phi}\,\sin\bar{\psi}}{\sin\bar{\phi}\,\sin\bar{\theta}\,\sin\bar{\psi}\;+\cos\bar{\phi}\,\cos\bar{\psi}}$$

$$-\tan\psi = \frac{\dfrac{\sin\bar{\phi}\,\sin\bar{\theta}\,\cos\psi}{\cos\bar{\phi}\,\cos\bar{\psi}} - \dfrac{\cos\bar{\phi}\,\sin\bar{\psi}}{\cos\bar{\phi}\,\cos\bar{\psi}}}{\dfrac{\sin\bar{\phi}\,\sin\bar{\theta}\,\sin\bar{\psi}}{\cos\bar{\phi}\,\cos\bar{\psi}} + \dfrac{\cos\bar{\phi}\,\cos\bar{\psi}}{\cos\bar{\phi}\,\cos\bar{\psi}}}$$

$$-\tan\psi = \frac{\tan\bar{\phi}\,\sin\bar{\theta} - \tan\bar{\psi}}{\tan\bar{\phi}\,\sin\bar{\theta}\,\tan\bar{\psi} + 1}$$

LET $\tan\bar{\phi}\,\sin\bar{\theta} = \tan X$

LET $\tan\bar{\psi} = \tan Y$

$$-\tan\psi = \frac{\tan X - \tan Y}{\tan X\,\tan Y + 1} = \tan(X-Y)$$

$$X = \tan^{-1}(\tan\bar{\phi}\,\sin\bar{\theta})$$

$$Y = \bar{\psi}$$

$$-\tan\psi = \tan\left[\tan^{-1}(\tan\bar{\phi}\,\sin\bar{\theta}) - \bar{\psi}\right]$$

$$-\psi = \tan^{-1}(\tan\bar{\phi}\,\sin\bar{\theta}) - \bar{\psi}$$

$$\psi = \tan^{-1}(\tan\bar{\phi}\,\sin\bar{\theta}) + \bar{\psi}$$

SUMMARIZING:

$$\bar{\phi} = \tan^{-1}(\sec\theta\,\tan\phi)$$

$$\bar{\theta} = \sin^{-1}(\sin\theta\,\cos\phi)$$

$$\bar{\psi} = \tan^{-1}(\tan\theta\,\sin\phi) + \psi \qquad M(\bar{\psi}, \bar{\theta}, \bar{\phi}) = M(\psi, \phi, \theta)$$

$$\phi = \sin^{-1}(\sin\bar{\phi}\,\cos\bar{\theta})$$

$$\theta = \tan^{-1}(\sec\bar{\phi}\,\tan\bar{\theta})$$

$$\psi = \bar{\psi} - \tan^{-1}(\tan\bar{\phi}\,\sin\bar{\theta})$$

CONVERSION BETWEEN $(\bar{\psi}, \bar{\theta}, \bar{\phi})$ AND (ϕ, ψ, θ)

EQUATING THE TWO MATRICES OF FIGURES 3-2 AND 303.

$$M(\bar{\psi}, \bar{\theta}, \bar{\phi}) = M(\phi, \psi, -\theta)$$

To find $\bar{\Psi}$:

$$\frac{B^*_{12}}{B^*_{11}} = \frac{B_{12}}{B_{11}}$$

$$\frac{c\,\bar{\theta}\,s\bar{\Psi}}{c\,\bar{\theta}\,c\psi} = \frac{c\,\theta\,s\,\psi c\phi + s\theta s\phi}{c\theta\,c\,\psi}$$

$$\tan\bar{\Psi} = \tan\psi\,c\,\phi + \tan\,\theta\,s\,\phi\,\sec\psi$$

$$\bar{\Psi} = \tan^{-1}(\tan\psi\cos\,\phi + \tan\,\theta\,\sin\,\phi\,\sec\psi\,)$$

To find $\bar{\theta}$: $\quad B^*_{13} = B_{13}$

$$-s\bar{\theta} = s\,\psi s\,\phi\,c\,\theta - s\,\theta\,c\,\phi$$
$$\bar{\theta} = -\sin^{-1}(\sin\psi\sin\,\phi\,\cos\,\theta - \sin\,\theta\,\cos\,\phi$$

To find $\bar{\Phi}$:

$$\frac{B^*_{33}}{B^*_{23}} = \frac{B_{33}}{B_{23}}$$

$$\frac{c\bar{\Phi}\,c\bar{\theta}}{s\bar{\Phi}\,c\bar{\theta}} = \frac{s\theta s\phi s\,\psi + c\theta c\phi}{c\,\psi s\,\phi}$$

$$\bar{\Phi} = \cot^{-1}(\sin\theta\,\tan\psi + \cos\,\theta\,\sec\psi\,\cot\,\phi)$$

To find ψ: $\quad B_{21} = B^*_{21}$

$$-s\,\psi = s\,\bar{\Phi}\,s\bar{\theta}\,c\bar{\Psi} - c\bar{\Phi}\,s\,\bar{\Psi}$$
$$= \sin^{-1}(\sin\,\phi\,\sin\,\theta\,\cos\bar{\Psi} - \cos\,\phi\,\sin\bar{\Psi}\,)$$

To find θ:

$$\frac{B_{31}}{B_{11}} = \frac{B^*_{31}}{B^*_{11}}$$

$$\frac{s\,\theta\,c\psi}{c\,\theta\,c\psi} = \frac{c\,\bar{\Phi}\,s\bar{\theta}c\,\bar{\Psi} + s\,\bar{\Phi}\,s\bar{\Psi}}{c\,\bar{\theta}\,c\,\bar{\Psi}}$$

$$\tan\,\theta = \frac{c\,\bar{\Phi}\,s\,\bar{\theta}\,c\,\bar{\Psi}}{c\,\bar{\theta}\,c\bar{\Psi}} + \frac{s\,\bar{\Phi}\,s\bar{\Psi}}{c\,\bar{\theta}c\psi}$$

$$\theta = \tan^{-1}(\cos\,\bar{\Phi}\,\tan\bar{\theta} + \sin\,\bar{\Phi}\,\sec\bar{\theta}\tan\bar{\Psi})$$

To find ϕ: $$\frac{B_{22}}{B_{23}} = \frac{B^*_{22}}{B^*_{23}}$$

$$\frac{c\phi c\psi}{s\phi c\psi} = \frac{s\bar{\phi}\, s\bar{\theta}\, s\bar{\psi} + c\bar{\phi} c\bar{\psi}}{s\bar{\phi}\, c\bar{\theta}}$$

$$\cot \phi = \frac{s\bar{\phi} s\, \theta\, s\bar{\psi}}{s\bar{\phi}\, c\bar{\theta}} + \frac{c\bar{\phi} c\bar{\psi}}{s\bar{\phi} c\bar{\theta}}$$

$$\phi = \cot^{-1}(\tan\bar{\theta}\, s\bar{\psi} + \cot\bar{\phi}\cos\bar{\psi}\sec\theta)$$

Summarizing:

$$\bar{\psi} = \tan^{-1}(\tan\psi\cos\phi + \tan\theta\sin\phi\sec\psi)$$

$$\bar{\theta} = -\sin^{-1}(\sin\psi\sin\phi\cos\theta - \sin\theta\cos\phi)$$

$$\phi = \cot^{-1}(\sin\theta\tan\psi + \cos\theta\sec\psi\cot\phi)$$

$$\psi = \sin^{-1}(\sin\bar{\phi}\sin\bar{\theta}\cos\bar{\psi} - \cos\bar{\phi}\sin\bar{\psi})$$

$$\theta = \tan^{-1}(\cos\bar{\phi}\tan\bar{\theta} + \sin\bar{\phi}\sec\bar{\theta}\tan\bar{\psi})$$

$$\phi = \cot^{-1}(\tan\bar{\theta}\sin\bar{\psi} + \cot\bar{\phi}\cos\bar{\psi}\sec\bar{\theta})$$

$$M(\bar{\psi}, \bar{\theta}, \bar{\phi}) = M(\phi, \theta, \psi)$$

To find $\bar{\psi}$: $$\frac{B^*_{12}}{B^*_{11}} \qquad \frac{B_{12}}{B_{11}}$$

$$\frac{\cos\bar{\theta}\sin\bar{\psi}}{\cos\bar{\theta}\cos\bar{\psi}} = \frac{\cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi}{\cos\theta\cos\psi}$$

$$\tan\bar{\psi} = \tan\theta\sin\phi + \tan\psi\sec\theta\cos\phi$$

$$\bar{\psi} = \tan^{-1}(\tan\theta\sin\phi + \tan\psi\sec\theta\cos\phi)$$

To find $\bar{\theta}$: $B^*_{13} = B_{13}$

$$-\sin\bar{\theta} = -\sin\theta\cos\phi\cos\psi + \sin\psi\sin\phi$$

$$\bar{\theta} = \sin^{-1}(\sin\theta\cos\phi\cos\psi - \sin\psi\sin\phi)$$

To find Φ: $$\frac{B^*_{23}}{B^*_{33}} = \frac{B_{23}}{B_{33}}$$

$$\frac{\sin \Phi \cos \Theta}{\cos \Phi \cos \Theta} \quad \frac{\sin \theta \sin\psi \cos \phi + \cos\psi \sin \phi}{\cos \theta \cos \phi}$$

$$\tan \Phi = \tan \theta \sin\psi + \cos\psi \tan \phi \sec \theta$$

$$\Phi = \tan^{-1}(\tan \theta \sin\psi + \cos\psi \tan \phi \sec \theta)$$

To find Θ: $B_{31} \quad B^*_{31}$

$$\sin \theta = \cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi$$

$$\Theta = \sin^{-1}(\cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi)$$

To find ϕ: $$\frac{B_{32}}{B_{33}} \quad \frac{B^*_{32}}{B^*_{33}}$$

$$\frac{-\sin \phi \cos \theta}{\cos \phi \cos \theta} = \frac{\cos \Phi \sin \Theta \sin \Psi - \sin \Phi \cos \Psi}{\cos \Theta \cos \Psi}$$

$$-\tan \phi = \tan \Phi \sec \Theta \cos \Psi + \tan \Theta \sin \Psi$$

$$\phi = \tan^{-1}(\tan \Phi \sec \Theta \cos \Psi + \tan \Theta \sin \Psi)$$

To find ψ: $$\frac{B_{21}}{B_{11}} \quad \frac{B^*_{21}}{B^*_{11}}$$

$$\frac{-\sin \psi \cos\theta}{\cos \psi \cos \theta} = \frac{\sin \Phi \sin \Theta \cos \Psi - \cos \Phi \sin \Theta}{\cos \Theta \cos \Psi}$$

$$\tan \psi = \cos \Phi \tan \Psi \sec \Theta - \sin \Phi \tan \Theta$$

$$\psi = \tan^{-1}(\cos \Phi \tan \Psi \sec \Theta - \sin \Phi \tan \Theta)$$

Summarizing:

$$\Psi = \tan^{-1}(\tan \theta \sin \phi + \tan\psi \sec \theta \cos \phi)$$

$$\Theta = \sin^{-1}(\sin \theta \cos \phi \cos \psi - \sin \psi \sin \phi)$$

$$\Phi = \tan^{-1}(\tan\theta \sin\psi + \cos\psi \tan\phi \sec\theta)$$
$$\theta = \sin^{-1}(\cos\Phi \sin\Theta \cos\Psi + \sin\Phi \sin\Psi)$$
$$\phi = \tan^{-1}(\tan\Phi \sec\Theta \cos\Psi + \tan\Theta \sin\Psi)$$
$$\psi = \tan^{-1}(\cos\Phi \tan\Psi \sec\Theta - \sin\Phi \tan\Theta)$$

(3-38) $M(\Psi, \Theta, \Phi) = M(\psi, \phi, \theta)$

CONVERSION BETWEEN (Ψ, Θ, Φ) AND (θ, ψ, ϕ)

EQUATING THE TWO MATRICES OF FIGURES 3-2 AND 3-5. $M(\Psi, \Theta, \Phi) = M(\theta, \psi, \phi)$.

TO FIND Ψ: $\dfrac{B^*_{12}}{B^*_{11}} = \dfrac{B_{12}}{B_{11}}$

$$\frac{\cos\Theta \sin\Psi}{\cos\Theta \cos\Psi} = \frac{\sin\psi}{\cos\psi \cos\theta}$$

$$\tan\Psi = \tan\psi \sec\theta$$

$$\Psi = \tan^{-1}(\tan\psi \sec\theta)$$

TO FIND Θ:

$$B^*_{13} = B_{13}$$

$$-\sin\Theta = -\cos\psi \sin\theta$$

$$\Theta = \sin^{-1}(\cos\psi \sin\theta)$$

TO FIND Φ:

$$\frac{B^*_{23}}{B^*_{33}} = \frac{B_{23}}{B_{33}}$$

$$\frac{\sin\Phi \cos\Theta}{\cos\Phi \cos\Theta} = \frac{\cos\phi \sin\psi \sin\theta + \sin\phi \cos\theta}{-\sin\phi \sin\psi \sin\theta + \cos\phi \cos\theta}$$

$$\tan\Phi = \frac{\dfrac{\cos\phi \sin\psi \sin\theta}{\cos\phi \cos\theta} + \dfrac{\sin\phi \cos\theta}{\cos\phi \cos\theta}}{\dfrac{\cos\phi \cos\theta}{\cos\phi \cos\theta} - \dfrac{\sin\phi \sin\psi \sin\theta}{\cos\phi \cos\theta}}$$

$$\tan \bar{\Phi} = \frac{\tan \theta \sin \psi + \tan \phi}{1 - \tan \phi \tan \theta \sin \psi}$$

LET: $\tan X = \tan \theta \sin \psi$; $\tan Y = \tan \phi$

$$X = \tan^{-1}(\tan\theta \sin \psi); Y = \phi$$

$$\tan \bar{\Phi} = \tan \left[\tan^{-1}(\tan \theta \sin \psi) + \phi \right]$$

$$\bar{\Phi} = \phi + \tan^{-1}(\tan \theta \sin \psi)$$

SUMMARIZING:

$$\bar{\Psi} = \tan^{-1}(\tan \psi \sec \theta)$$

$$\bar{\Theta} = \sin^{-1}(\cos \psi \sin \theta)$$

$$\bar{\Phi} = \phi + \tan^{-1}(\tan \theta \sin \psi) \qquad (3\text{-}39)$$

$$\psi = \sin^{-1}(\cos \bar{\Theta} \sin \bar{\Psi}) \qquad M(\bar{\Psi}, \bar{\Theta}, \bar{\Phi}) = M(\theta, \psi, \phi)$$

$$\theta = \tan^{-1}(\tan \bar{\Theta} \sec \bar{\Psi})$$

$$\phi = \bar{\Phi} - \tan^{-1}(\sin \bar{\Theta} \tan \bar{\Psi})$$

NOTE: THE VALUES FOR θ, ψ AND ϕ WERE OBTAINED IN "PROBLEM EXAMPLE" PARAGRAPHS 3-30, 3-31, AND 3-34.

CONVERSION BETWEEN ($\bar{\Psi}$, $\bar{\Theta}$, $\bar{\Phi}$) AND (θ, ϕ, ψ)

EQUATING THE TWO MATRICES OF FIGURES 3-2 AND 3-6.

$$M(\bar{\Psi}, \bar{\Theta}, \bar{\Phi}) = M(\theta, \phi, \psi)$$

TO FIND $\bar{\Theta}$:

$$B^*_{13} = B_{13}$$

$$-s\bar{\Theta} = s\psi\, s\phi\, c\,\theta - c\psi\, s\theta$$

$$\bar{\Theta} = \sin^{-1}(c\psi\, s\theta - s\psi\, s\,\phi\, c\,\phi)$$

$$\bar{\Theta} = \sin^{-1}(\cos\psi \sin\theta - \sin\psi \sin\phi \cos\theta)$$

$\bar{\Psi}$: $\dfrac{B^*_{11}}{B^*_{12}} = \dfrac{B_{11}}{B_{12}}$

$$\frac{c\cancel{\theta}c\bar{\Psi}}{c\cancel{\theta}\,\psi} = \frac{s\psi\, s\phi\, s\theta + c\psi\, c\theta}{s\psi\, c\,\phi}$$

$$\cot\bar{\Psi} = \tan\phi\, s\theta + \cot\psi c\theta \sec\phi$$

$$\bar{\Psi} \quad \cot^{-1}(\tan\phi \sin\theta + \cot\psi \cos\theta \sec\phi)$$

$\bar{\Phi}$: $\dfrac{B^*_{23}}{B^*_{33}} = \dfrac{B_{23}}{B_{33}}$

$$\frac{s\bar{\Phi}c\ O}{c\bar{\Phi}c\ O} = \frac{c\psi\, s\phi c\theta + s\psi s\theta}{c\phi\ c\theta}$$

$$\tan\bar{\Phi} = \cos\psi \tan\phi + \sin\psi \sec\phi \tan\theta$$

$$\bar{\Phi} = \tan^{-1}(\cos\psi \tan\phi + \sin\psi \sec\phi \tan\theta)$$

ϕ : $B_{32} = B^*_{32}$

$$-s\phi = c\,\bar{\Phi}\, s\cancel{\theta} s\psi - s\,\bar{\Phi}c\bar{\Psi}$$

$$\phi = \sin^{-1}(\sin\bar{\Phi}\cos\bar{\Psi} - \cos\bar{\Phi}\sin\bar{\Psi}\sin\cancel{\theta})$$

ψ : $\dfrac{B_{22}}{B_{12}} = \dfrac{B^*_{22}}{B^*_{12}}$

$$\frac{c\,\psi\, c\phi}{s\psi\ c\phi} = \frac{s\bar{\Phi}s\ O\ s\bar{\Psi} + c\,\bar{\Phi}\ c\bar{\Psi}}{c\cancel{\theta}s\bar{\Psi}}$$

$$\cot\psi = \sin\bar{\Phi}\tan\cancel{\theta} + \cos\bar{\Phi}\sec\cancel{\theta}\cot\bar{\Psi}$$

$$\psi = \cot^{-1}(\sin\bar{\Phi}\tan\cancel{\theta} + \cos\bar{\Phi}\sec\cancel{\theta}\cot\bar{\Psi})$$

θ : $\dfrac{B_{31}}{B_{33}} = \dfrac{B^*_{31}}{B^*_{33}}$

$$\frac{c\phi s\theta}{c\phi c\theta} = \frac{c\bar{\Phi}s\cancel{\theta}c\bar{\Psi} + s\bar{\Phi}s\bar{\Psi}}{c\bar{\Phi}c\cancel{\theta}}$$

$$\tan\theta = \tan\cancel{\theta}\cos\bar{\Psi} + \tan\bar{\Phi}\sin\bar{\Psi}\sec\cancel{\theta}$$

$$\theta = \tan^{-1}(\tan\cancel{\theta}\cos\bar{\Psi} + \tan\bar{\Phi}\sin\bar{\Psi}\sec\cancel{\theta}$$

SUMMARIZING:

$$
\begin{aligned}
\bar{\Theta} &= \sin^{-1}(\cos\psi \sin\theta - \sin\psi \sin\phi \cos\theta) \\
\bar{\Psi} &= \cot^{-1}(\tan\phi \sin\theta + \cot\psi \cos\theta \sec\phi) \\
\bar{\Phi} &= \tan^{-1}(\cos\psi \tan\phi + \sin\psi \sec\phi \tan\theta) \\
\phi &= \sin^{-1}(\sin\bar{\Phi} \cos\bar{\Psi} - \cos\bar{\Phi} \sin\bar{\Theta} \sin\bar{\Psi}) \\
\psi &= \cot^{-1}(\sin\bar{\Phi} \tan\bar{\Theta} + \cos\bar{\Phi} \sec\bar{\Theta} \cot\bar{\Psi}) \\
\theta &= \tan^{-1}(\tan\bar{\Theta} \cos\bar{\Psi} + \tan\bar{\Phi} \sin\bar{\Psi} \sec\bar{\Theta})
\end{aligned}
\qquad (3\text{-}40)
$$

$$M(\quad, 0, \phi) = M(\theta, \phi, \quad)$$

TRANSFORMATIONS FROM DIFFERENTIAL EQUATIONS IN THE DIRECTION COSINES (WITHOUT EULER ANGLES)

THE DIRECTION COSINES CAN BE OBTAINED DIRECTLY IF THE COMPONENTS OF THE INERTIAL ANGULAR VELOCITIES OF THE TWO FRAMES ARE KNOWN. THIS METHOD FOR THE DETERMINATION OF THE TRANSFORMATION MATRIX DOES NOT REQUIRE KNOWLEDGE OF THE EULER ANGLES, NOR A PARTICULAR SEQUENCE, BUT THE INITIAL VALUES OF THE DIRECTION COSINES MUST BE SPECIFIED. THE DIRECTION COSINES BETWEEN THE $\overline{R}_I$ FRAME AND $\overline{F}_I$ FRAME ARE REPRESENTED BY THE MATRIX EQUATION:

$$\begin{pmatrix} \overline{R}_1 \\ \overline{R}_2 \\ \overline{R}_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \overline{F}_1 \\ \overline{F}_2 \\ \overline{F}_3 \end{pmatrix} \qquad (4\text{-}1)$$

BY EQUATION (B)

$$\overline{R}_J = A_{JL}\overline{F}_1 + A_{J2}\overline{F}_2 + A_{J3}\overline{F}_3 \qquad (4\text{-}2)$$

AND HENCE

$$\dot{\overline{R}}_J = \dot{A}_{JL}\overline{F}_1 + \dot{A}_{J2}\overline{F}_2 + \dot{A}_{J3}\overline{F}_3 + A_{JL}\dot{\overline{F}}_1 \qquad (4\text{-}3)$$

$$+ A_{J2}\dot{\overline{F}}_2 + A_{J3}\dot{\overline{F}}_3$$

THE TIME DERIVATIVES OF $\overline{F}_J$ ARE GIVEN BY

$$\dot{\overline{F}}_1 = \frac{\delta \dot{\overline{F}}_1}{\delta T} + \overline{\omega}_F \times \overline{F}_J \qquad J = 1,\ 2,\ 3 \qquad (4\text{-}4)$$

SINCE

$$\frac{\delta \dot{\overline{F}}_J}{\delta T} = 0.$$

EQUATION (4-4) BECOMES

$$\dot{\overline{F}}_J = \overline{\omega}_F \times \overline{F}_J \qquad J = 1,\ 2,\ 3.$$

OR

$$\dot{\overline{F}}_1 = R_F\overline{F}_2 - Q_F\overline{F}_3 \qquad (4\text{-}5)$$

$$\dot{\overline{F}}_2 = P_F\overline{F}_3 - R_F\overline{F}_1 \qquad (4\text{-}6)$$

$$\dot{\overline{F}}_3 = Q_F\overline{F}_1 - P_F\overline{F}_2 \qquad (4\text{-}7)$$

THE TIME DERIVATIVES OF THE $\overline{R}_i$ ARE

$$\dot{\overline{R}}_1 = R_R\overline{R}_2 - Q_R\overline{R}_3 \tag{4-8}$$

$$\dot{\overline{R}}_2 = P_R\overline{R}_3 - R_R\overline{R}_1 \tag{4-9}$$

$$\dot{\overline{R}}_3 = Q_R\overline{R}_1 - P_R\overline{R}_2 \tag{4-10}$$

UTILIZING EQUATIONS (4-5), (4-6), (4-7) AND (4-8) WITH EQUATIONS (4-3), $J = 1$, AND COLLECTING LIKE COMPONENTS

$$\begin{aligned} R_R\overline{R}_2 - Q_R\overline{R}_3 = {} & (\dot{A}_{11} - A_{12}R_F + A_{13}Q_F)\overline{F}_1 \\ & + (\dot{A}_{12} - A_{13}P_F + A_{11}R_F)\overline{F}_2 \\ & + (\dot{A}_{13} - A_{11}Q_F + A_{12}P_F)\overline{F}_3 \end{aligned} \tag{4-11}$$

DOTTING (4-11) WITH $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$, RESPECTIVELY, AND USING EQUATION (4-1) THE FOLLOWING DIFFERENTIAL EQUATIONS ARE OBTAINED:

$$\begin{aligned} \dot{A}_{11} &= A_{12}R_F - A_{13}Q_F + A_{21}R_R - A_{31}Q_R \\ \dot{A}_{12} &= A_{13}P_F - A_{11}R_F + A_{22}R_R - A_{32}Q_R \\ \dot{A}_{13} &= A_{11}Q_F - A_{12}P_F + A_{23}R_R - A_{33}Q_R \end{aligned} \tag{4-12}$$

IN A SIMILAR MANNER

$$\begin{aligned} P_R\overline{R}_3 - R_R\overline{R}_1 = {} & (\dot{A}_{21} - A_{22}R_F + A_{23}Q_F)\overline{F}_1 \\ & + (\dot{A}_{22} - A_{23}P_F + A_{21}R_F)\overline{F}_2 + \\ & (\dot{A}_{23} - A_{21}Q_F + A_{22}P_F)\overline{F}_3, \end{aligned} \tag{4-13}$$

AND

$$\begin{aligned} Q_R\overline{R}_1 - P_R\overline{R}_2 = {} & (\dot{A}_{31} - A_{32}R_F + A_{33}Q_F)\overline{F}_1 \\ & + (\dot{A}_{32} - A_{33}P_F + A_{31}R_F)\overline{F}_2 + \\ & (\dot{A}_{33} - A_{31}Q_F + A_{32}P_F)\overline{F}_3 \end{aligned} \tag{4-14}$$

ARE OBTAINED. DOTTING (4-13) AND (4-14) WITH F_1, F_2, AND F_3, RESPECTIVELY,

AND USING EQUATION (4-1), THE FOLLOWING TWO SETS OF EQUATIONS ARE OBTAINED:

$$\dot{A}_{21} = A_{22}R_F - A_{23}Q_F + A_{31}P_R - A_{11}R_R$$
$$\dot{A}_{22} = A_{23}P_F - A_{21}R_F + A_{32}P_R + A_{12}R_2 \qquad (4\text{-}15)$$
$$\dot{A}_{23} = A_{21}Q_F - A_{22}P_F + A_{33}P_R - A_{13}R_R$$

AND

$$\dot{A}_{31} = A_{32}R_F - A_{33}Q_F + A_{11}Q_R - A_{21}P_R$$
$$\dot{A}_{32} = A_{33}P_F - A_{31}R_F + A_{12}Q_R - A_{22}P_R$$
$$\dot{A}_{33} = A_{31}Q_F - A_{32}P_F + A_{13}Q_R - A_{23}P_R \qquad (4\text{-}16)$$

THE NINE DIFFERENTIAL EQUATIONS GIVEN IN EQUATIONS (4-12), (4-13), AND (4-14) CAN BE WRITTEN AS THE SINGLE EQUATION:

$$\dot{A}_{JI} = A_{J(I+1)}\,\omega_{F(I+2)} - A_{J(I+2)}$$
$$\omega_{F(I+1)} + A_{(J+1)}$$
$$\omega_{R(J+2)} - A_{(J+2)I}\,\omega_{R(J+1)}. \qquad (4\text{-}17)$$
$$(I = 1,2,3,)$$
$$(J = 1,2,3).$$

THE CONSTRAINING EQUATIONS ON THE SOLUTIONS OF (4-17) ARE, OF COURSE, THOSE GIVEN IN B-12. THESE SIX RELATIONS WILL BE IDENTICALLY SATISFIED BY THE SOLUTIONS OF 4-17. IT IS TO BE NOTED THAT THE EQUATIONS (B-10):

$$\sum_{I=1}^{3} A_{JI}^2 = 1; \quad J = 1,2,3$$

$$\sum_{I=1}^{3} A_{JI}A_{(J+1)I} = 0; \quad J = 1,2,3,$$

CAN BE USED TO REDUCE THE SET OF NINE DIFFERENTIAL EQUATIONS, (4-17), TO A SET OF THREE DIFFERENTIAL EQUATIONS. FOR SUPPOSE A_{11}, A_{12}, AND A_{23} ARE CHOSEN AS INDEPENDENT, IT IS THEN POSSIBLE TO SOLVE THE EQUATIONS (4-18) FOR A_{13}, A_{21}, A_{22}, A_{31}, A_{32}, AND A_{33} AS FUNCTIONS OF A_{11}, A_{12}, AND A_{23}. EQUATION (4-17) WILL THEN REDUCE TO THREE EQUATIONS WHICH INVOLVE $\dot{A}_{11}$, $\dot{A}_{12}$, $\dot{A}_{23}$, A_{11}, A_{12}, A_{23}, $P_F Q_F R_F$ AND $P_R Q_R R_R$.

THE SOLUTIONS OF THE DIFFERENTIAL EQUATION GIVEN IN (4-17) ARE, OF COURSE, THE DIRECTION COSINES A_{JI}. IT IS NOTED THAT $P_F Q_F R_F$ AND $P_R Q_R R_R$, AS WELL AS INITIAL VALUES ON A_{JI}, MUST BE SPECIFIED IN ORDER THAT THE EQUATIONS HAVE SOLUTIONS.

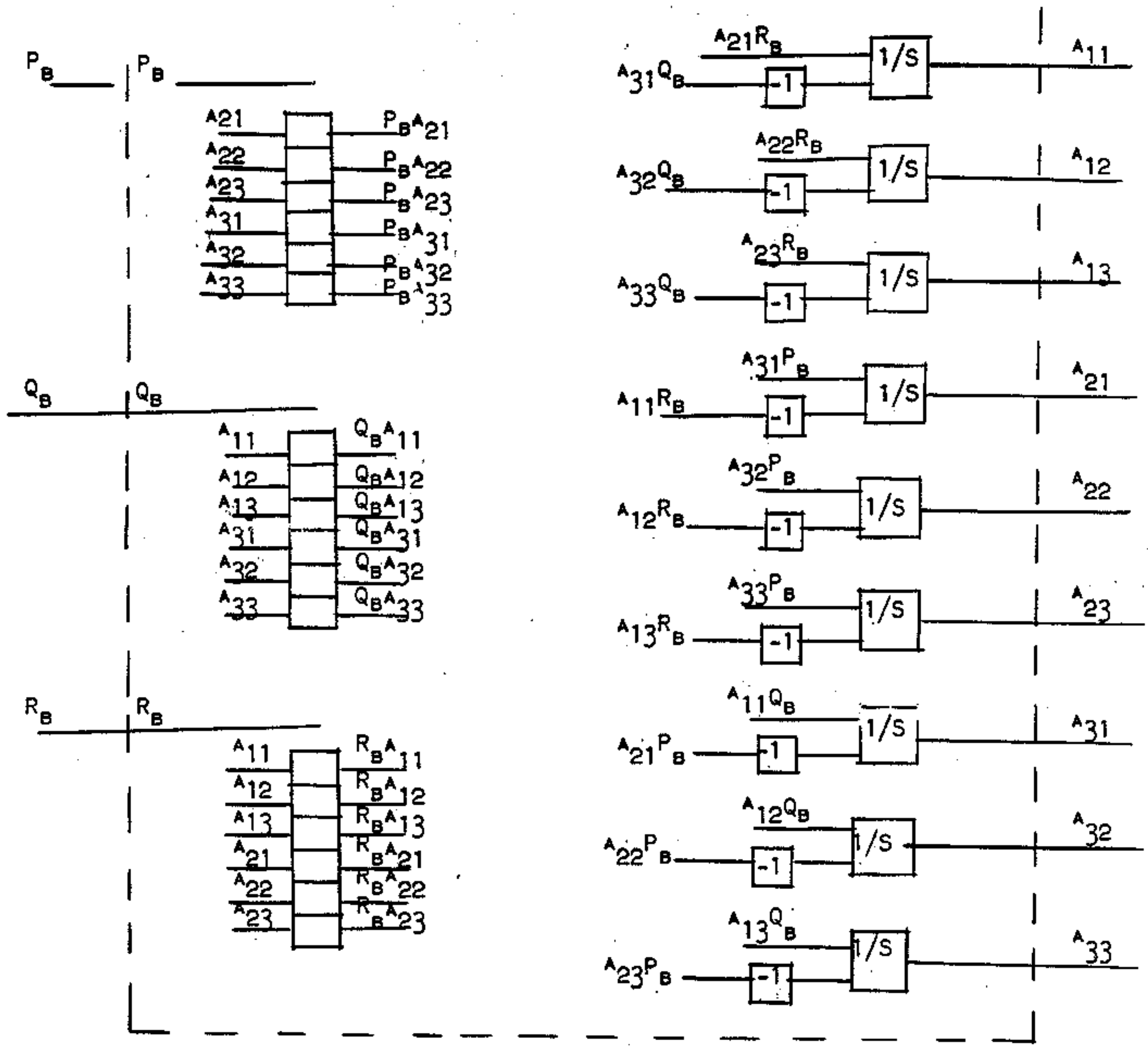


FIG. 4-1. MECHANIZATION OF DIRECTION COSINES USING DERIVATIVE OF DIRECTION COSINE TECHNIQUE.

TRANSFORMATION BETWEEN TWO REFERENCE FRAMES AS THE PRODUCT OF TWO DIRECTION COSINE MATRICES WHEN DIRECTION COSINE MATRICES BETWEEN EACH OF THE TWO REFERENCE FRAMES AND A THIRD REFERENCE FRAME ARE KNOWN

[EXAMPLE: WIND AXES TO BODY AXES TO GROUND AXES.]

THIS METHOD OF OBTAINING THE DIRECTION COSINE MATRIX BETWEEN TWO REFERENCE FRAMES IS THE THIRD METHOD OF THE THREE COMMONLY USED METHODS MENTIONED IN SECTION 1. A PARTICULAR EXAMPLE IS TAKEN TO ILLUSTRATE THE METHOD.

CONSIDER THE ORIENTATION OF THE WIND AXES $\overline{W}$, WITH RESPECT TO THE GROUND REFERENCE FRAME $\overline{G}$, AS SHOWN IN FIGURE 5-1 THROUGH THE EULER SEQUENCE YAW, PITCH, ROLL WHERE THE YAW, PITCH AND ROLL MATRICES ARE DEFINED AS FOLLOWS

$$
\begin{aligned}
&M_Y(\psi_V)\\
&M_P(\gamma)\\
&M_R = (\phi_V)
\end{aligned}
\qquad (5\text{-}1)
$$

FROM EQUATION (5-1) IT IS SEEN THAT THE NORMAL PROJECTION OF THE VELOCITY VECTOR (ABSOLUTE VELOCITY WHEN NO EXTERNAL WINDS, OR RELATIVE VELOCITY WITH EXTERNAL WINDS) IN THE HORIZONTAL PLANE MAKES AN ANGLE ψ_V WITH RESPECT TO THE $\overline{G}_1$ AXES, THE PITCH ANGLE γ IS THE ANGLE MEASURED IN THE VERTICAL PLANE WHICH THE "VELOCITY VECTOR" MAKES WITH THE HORIZONTAL.

THE ORIENTATION OF THE WIND AXES WITH RESPECT TO THE GROUND AXES IS

$$\overline{W}_I = M_R(\phi_V)M_P(\gamma)M_Y(\psi_V)\overline{G}_J \qquad (5\text{-}2)$$

THE ORIENTATION OF THE BODY AXES WITH RESPECT TO THE WIND AXES THROUGH POSITIVE ANGLES AS SHOWN IN FIGURE 5-2 IS

$$\overline{B}_I = M_P(\alpha)M_Y(\beta)\overline{W}_J \qquad (5\text{-}3)$$

OR THE INVERSE OF (5-3)

$$\overline{W}_I = M_Y(-\beta)M_P(-\alpha)\overline{B}_J, \qquad (5\text{-}4)$$

OR

$$
\begin{pmatrix}\overline{W}_1\\ \overline{W}_2\\ \overline{W}_3\end{pmatrix}
=
\begin{pmatrix}
c\beta\, c\alpha & -s\beta & c\beta\, s\alpha\\
s\beta\, c\alpha & c\beta & s\beta\, s\alpha\\
-s\alpha & 0 & c\alpha
\end{pmatrix}
\begin{pmatrix}\overline{B}_1\\ \overline{B}_2\\ \overline{B}_3\end{pmatrix}
\qquad (5\text{-}5)
$$

The orientation of the body axes B_i with respect to the ground axes for the sequency (ψ, θ, ϕ) is

$$\bar{B}_i = M(\psi, \theta, \phi)\,\bar{G}_j \tag{5-6}$$

By equation (5-5) and (5-6) and Figure 3-2

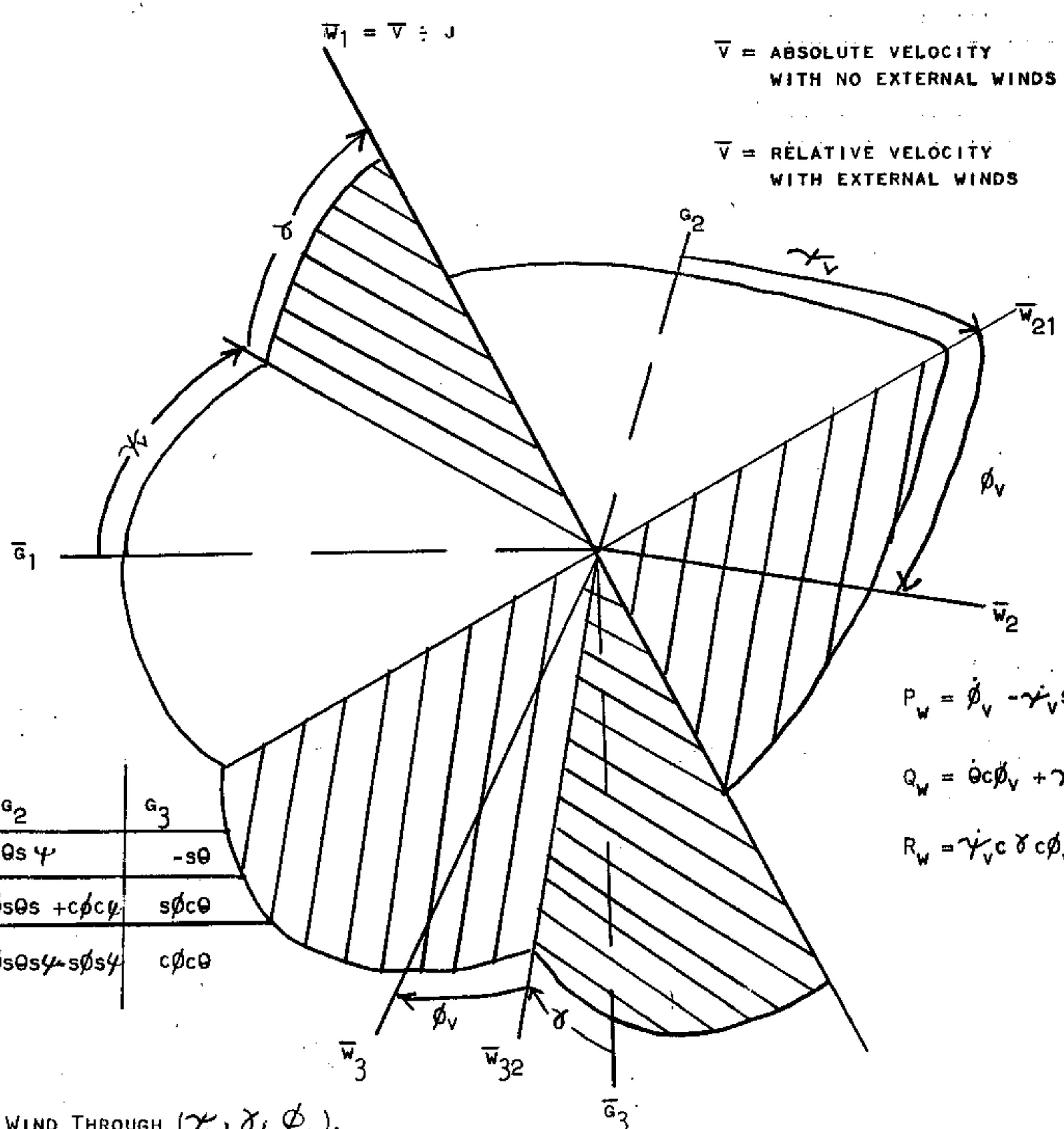
$$\bar{W}_i = M(-\alpha, -\beta)M(\psi, \theta, \phi)\bar{G}_j$$

or

$$\begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \end{pmatrix} = \begin{pmatrix} c\beta\, c\alpha & -s\beta & c\beta\, s\alpha \\ s\beta\, c\alpha & c\beta & s\beta\, s\alpha \\ -s\alpha & 0 & c\alpha \end{pmatrix} \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\theta\psi & c\phi c\theta \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} \tag{5-7}$$

This equation (5-7) gives the direction cosines from ground axes to wind axes through 5 angles.

The common reference frame eliminated in this case is the body axis.



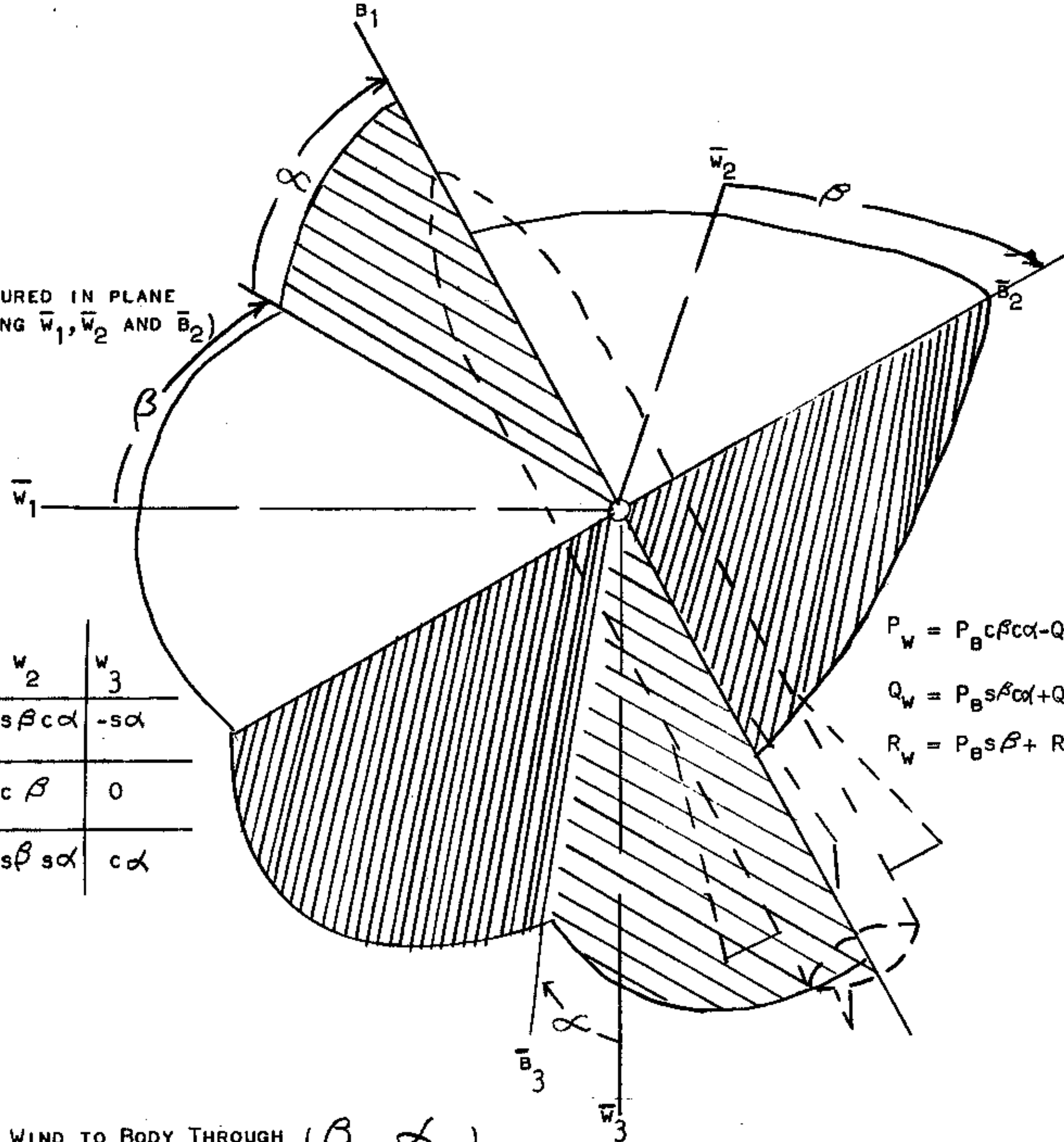
$$P_w = \dot{\phi}_v - \dot{\psi}_v s \quad + K_1$$

$$Q_w = \dot{\theta} c\phi_v + \dot{\psi}_v s\phi_v c\,\gamma + K_2$$

$$R_w = \dot{\psi}_v c\,\gamma\, c\phi_v - \dot{\theta} s\phi_v + K_3$$

	G_1	G_2	G_3
W_1	$c\theta c\psi$	$c\theta s\psi$	$-s\theta$
W_2	$s\phi s\theta c\psi - c\phi s\psi$	$s\phi s\theta s + c\phi c\psi$	$s\phi c\theta$
W_3	$c\phi s\theta c\psi + s\phi s\psi$	$c\phi s\theta s\psi - s\phi s\psi$	$c\phi c\theta$

FIG. 5- GROUND TO WIND THROUGH (ψ_v, γ, ϕ_v).



	W_1	W_2	W_3
B_1	$c\beta c\alpha$	$s\beta c\alpha$	$-s\alpha$
B_2	$-s\beta$	$c\beta$	0
B_3	$c\beta s\alpha$	$s\beta s\alpha$	$c\alpha$

$$P_W = P_B c\beta c\alpha - Q_B s\beta + R_B c\beta s\alpha + \dot{\alpha} s\beta$$

$$Q_W = P_B s\beta c\alpha + Q_B c\beta + R_B s\beta s\alpha - \dot{\alpha} c\beta$$

$$R_W = P_B s\beta + R_B c\alpha - \dot{\beta}$$

FIG. 5-2. WIND TO BODY THROUGH (β, α)

Wind Axes (ψ_V, γ) as Functions of (ψ , θ, ϕ, α, β) for All body Attitude Sequences

For the ground to body sequence (θ, ϕ, ψ);

$$\bar{B}_i = M_Y(\psi)\, M_R(\phi) M_P(\theta)\, \bar{G}_i$$

and for wind to body sequence (β , α)

$$\bar{B}_i = M_P(\alpha)\, M_Y(\beta) \bar{W}_i$$

or $\bar{W}_i = M_Y(-\beta)\, M_P(-\alpha) \bar{B}_i$

$$\bar{W}_i = M_Y(-\beta) M_P(-\alpha) M_Y(\psi) M_R(\phi) M_P(\theta) \bar{G}_i$$

but since for the ground to wind sequence (ψ_V, γ)

$$\bar{W}_i = M_P(\gamma) M_Y(\psi_V) \bar{G}_i$$

$$M_P(\gamma) M_Y(\psi_V) = M_Y(-\beta) M_P(-\alpha) M_Y(\psi) M_P(\phi) M_P(\theta)$$

which is expressed as

$$\begin{pmatrix} c\gamma & 0 & -s\gamma \\ 0 & 1 & 0 \\ s\gamma & 0 & c\gamma \end{pmatrix} \begin{pmatrix} c\psi_V & s\psi_V & 0 \\ -s\psi_V & c\psi_V & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{pmatrix}$$

$$\begin{pmatrix} s\psi s\phi s\theta + c\psi c\theta & s\psi c\phi & s\psi s\phi c\theta - c\psi s\theta \\ c\psi s\phi s\theta - s\psi c\theta & c\psi c\phi & c\psi s\phi s\theta + s\psi s\theta \\ c\phi s\theta & -s\phi & c\phi c\theta \end{pmatrix} \quad (5\text{-}8)$$

WHICH YIELDS:

$$w_{11} = c\beta\, c\alpha\,(s\psi\, s\phi s\theta + c\,\psi c\theta) + s\,\beta(s\psi\, c\theta - c\psi\, s\phi s\theta) + c\beta\, s\alpha\, c\phi s\theta$$

$$w_{12} = c\beta\, c\alpha\, s\psi\, c\phi - s\beta\, c\psi\, c\phi - s\phi\, c\beta\, s\alpha$$

$$w_{13} = c\beta\, c\alpha\,(s\,\psi\, s\phi c\theta - c\psi\, s\theta) - s\beta\,(c\,\psi\, s\phi s\theta + s\,\psi\, s\theta) + c\beta\, s\alpha\, c\phi c\theta$$

$$w_{21} = s\beta\, c\alpha\,(s\psi\, s\phi s\theta + c\,\psi\, c\theta) + c\,\beta\,(c\,\psi\, s\phi s\theta - s\,\psi\, c\theta) + s\beta\, s\alpha\, c\phi s\theta$$

$$w_{22} = s\,\beta\, c\alpha\, s\psi\, c\phi + c\beta\, c\,\psi\, c\phi - s\beta\, s\alpha\, s\phi$$

$$w_{23} = s\beta\, c\alpha\,(s\psi\, s\phi c\theta - c\,\psi\, s\theta) + c\beta\,(c\psi\, s\phi s\theta + s\psi\, s\theta) + s\beta\, s\alpha\, c\phi c\theta$$

$$w_{31} = s\alpha\,(s\,\psi\, s\phi s\theta + c\,\psi c\theta + c\alpha\, c\phi s\theta$$

$$w_{32} = s\alpha\, s\,\psi\, c\phi - s\phi c$$

$$w_{33} = s\alpha\,(s\,\psi s\phi c\theta - c\,\psi\, s\theta) + c\alpha\, c\phi c\theta$$

WHERE w_{ij} $\begin{pmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{pmatrix}$ IS THE WIND TO GROUND MATRIX.

EQUATING ELEMENTS OF THE MATRICES THEN GIVES US:

$$c\,\gamma = s\alpha\,(c\,\psi\, s\theta - s\,\psi\, s\phi c\theta) + c\alpha\, c\phi c\theta$$

$$c\,\gamma = -s\alpha\, B_{13} + c\alpha\, B_{33} \tag{5-9}$$

$$c\,\gamma = w_{33}$$

$(\theta,\ \phi, \psi\ )$

$$c\,\psi_v = s\beta\, c\alpha\, s\,\psi\, c\phi + c\,\beta\, c\psi\, c\phi - s\beta\, s\alpha\, s\phi$$

$$c\,\psi_v = s\,\beta\, c\alpha\, B_{12} + c\,\beta\, B_{22} + s\,\beta s\alpha\, B_{32}$$

$$c\,\psi_v = w_{22}$$

WHERE B_{ij} $\begin{pmatrix} i = 1, 2,3 \\ j = 1, 2,3 \end{pmatrix}$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

FOR THE GROUND TO BODY SEQUENCE $(\phi,\ \theta, \psi\ )$, THE GROUND TO WIND MATRIX IS:

$$w_{11} = c\beta\, c\alpha\, c\theta c\psi + s\,\beta\, s\psi\, c\theta + c\beta\, s\alpha\, s\theta$$

$$w_{12} = c\beta\, c\alpha\, c\psi s\theta s\phi + c\beta\, c\alpha\, s\,\psi\, c\phi + s\,\beta\, s\psi\, s\theta s\phi - s\beta\, c\,\psi\, c\phi - c\,\beta\, s\alpha\, s\phi c\theta$$

$$w_{13} = -c\beta\, c\alpha\, s\theta c\phi c\psi + -c\beta\, c\alpha\, s\psi\, s\phi - s\beta\, s\theta s\psi\, c\phi - s\beta c\psi\, s\phi + c\beta\, s\alpha\, c\theta c\phi$$

$$w_{21} = s\beta\, c\alpha\, c\theta c\psi - c\beta s\psi\, c\theta + s\beta\, s\alpha s\theta$$

$$w_{22} = s\beta\, c\alpha\, c\psi\, s\theta s\phi + s\beta\, c\alpha\, s\psi\, c\phi - c\beta\, s\psi\, s\theta s\phi + c\beta\, c\psi\, c\phi - s\beta\, s\alpha\, s\phi c\theta$$

$$w_{23} = -s\beta\, c\alpha\, s\theta c\phi c\psi + s\beta\, c\alpha\, s\psi\, s\phi + c\beta s\theta s\psi c\phi + c\beta\, c\psi\, s\phi + s\beta\, s\alpha\, c\theta c\phi$$

$$w_{31} = -s\alpha\, c\theta c\psi + c\alpha\, s\theta$$

$$w_{32} = -s\alpha\, c\psi\, s\theta s\phi - s\alpha\, s\psi\, c\phi - c\alpha s\phi c\theta$$

$$w_{33} = s\alpha s\theta c\phi c\psi - s\alpha\, s\psi\, s\phi + c\alpha\, c\theta c\phi$$

WHERE $W_{IJ} \left(\begin{matrix} I \\ J \end{matrix} \right\} = 1,\ 2,\ 3 \Big)$ IS THE GROUND TO WIND MATRIX,

WHICH GIVES US:

$$c\,\gamma = s\alpha\,(s\theta c\phi c\psi - s\,\psi s\phi) + c\alpha\, c\theta c\phi$$

$$c\,\gamma = -s\alpha\, B_{13} + c\alpha\, B_{33} \tag{5-10}$$

$$c\,\gamma = W_{33}$$

$(\phi,\ \theta, \psi\ )$

$$c\,\psi_V = s\beta\, c\alpha\,(c\theta c\psi + s\,\psi\, c\phi) + c\beta\,(-s\psi\, s\theta s\phi + c\psi\, c\phi) - s\beta\, s\alpha s\phi c\theta$$

$$= s\beta c\alpha\, B_{12} + c\beta B_{22} + s\beta s\alpha\, B_{32}$$

$$= W_{22}$$

WHERE $B_{IJ} \left(\begin{matrix} I \\ J \end{matrix} \right\} = 1,\ 2, 3 \Big)$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

FOR THE GROUND TO BODY SEQUENCE $(\theta, \psi,\ \phi)$, THE GROUND TO WIND MATRIX IS:

$$w_{11} = c\beta\, c\alpha\, c\,\psi c\theta + s\beta\, c\phi s\psi\, c\theta - s\beta\, s\phi s\theta + c\beta\, s\alpha\, s\phi s\psi\, c\theta + c\beta\, s\alpha\, c\phi s\theta$$

$$w_{12} = c\beta c\alpha s\psi - s\beta\, c\phi c\psi - c\beta\, s\alpha\, s\phi c$$

$$w_{13} = -c\beta\, c\alpha\, c\,\psi s\theta - s\beta c\phi s\psi s\theta - s\beta\, s\phi c\theta - c\beta\, s\alpha\, s\phi s\psi\, s\theta + c\beta\, s\alpha\, c\phi c\theta$$

$$w_{21} = s\beta\, c\alpha\, c\,\psi\, c\theta + c\beta\, c\phi s\psi\, c\theta + c\beta\, s\phi s\theta + s\beta\, s\alpha s\phi s\psi\, c\theta + s\beta\, s\alpha c\phi s\theta$$

$$w_{22} = s\beta c\alpha s\psi + c\beta c\phi c\psi - s\beta s\alpha s\phi c\psi$$

$$w_{23} = -s\beta c\alpha c\psi s\theta + c\beta c\phi s\psi s\theta + c\beta s\phi c\theta - s\beta s\alpha s\phi s\psi s\theta + s\beta s\alpha c\phi c\theta$$

$$w_{31} = s\alpha c\psi c\theta + c\alpha s\phi s\psi c\theta + c\alpha c\phi s\theta$$

$$w_{32} = s\alpha s\psi - c\alpha s\phi c\psi$$

$$w_{33} = s\alpha c\psi s\theta - c\alpha s\phi s\psi s\theta + c\alpha c\phi c\theta$$

WHERE $w_{ij} \left(\begin{matrix} i \\ j \end{matrix} \right\} = 1, 2, 3)$ IS THE GROUND TO WIND MATRIX,

WHICH GIVES US:

$$\begin{aligned} c\gamma &= s\alpha(c\psi s\theta) + c\alpha(-s\phi s\theta s\psi + c\phi c\theta) \\ &= -s\alpha B_{13} + c\alpha B_{33} \qquad (5\text{-}11) \\ &= w_{33} \end{aligned}$$

(θ, ψ, ϕ)

$$\begin{aligned} c\psi_V &= s\beta c\alpha s\psi + c\beta c\phi c\psi - s\beta s\alpha s\phi c \\ &= s\beta c\alpha B_{12} + c\beta B_{22} + s\beta s\alpha B_{32} \\ &= w_{22} \end{aligned}$$

WHERE $B_{ij} \left(\begin{matrix} i \\ j \end{matrix} \right\} = 1, 2, 3)$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

FOR THE GROUND TO BODY SEQUENCE (ϕ, ψ, θ):

$$w_{11} = c\beta c\alpha c\theta c\psi + s\beta s\psi + c\beta s\alpha s\theta c\psi$$

$$w_{12} = c\beta c\alpha c\theta s\psi c\phi + c\beta c\alpha s\theta s\phi - s\beta c\phi c\psi + c\beta s\alpha s\theta s\psi c\phi - c\beta s\alpha c\theta s\phi$$

$$w_{13} = c\beta c\alpha s\psi s\phi c\theta - c\beta c\alpha s\theta c\phi - s\beta c\psi s\phi + c\beta s\alpha s\theta s\phi + c\beta s\alpha c\theta c\phi$$

$$w_{21} = s\beta c\alpha c\theta c\psi - c\beta s\psi + s\beta s\alpha s\theta c\psi$$

$$w_{22} = s\beta c\alpha s\theta s\psi c\phi + s\beta c\alpha s\theta s\phi + c\beta c\phi c\psi + s\beta s\alpha s\theta s\psi c\phi - s\beta s\alpha c\theta s\phi$$

$$w_{23} = s\beta c\alpha s\psi s\phi c\theta - s\beta c\alpha s\theta c\phi + c\beta c\psi s\phi + s\beta s\alpha s\theta s\psi + s\beta s\alpha c\theta c\phi$$

$$w_{31} = s\alpha c\theta c\psi + c\alpha s\theta c\psi$$

$$W_{32} = -s\alpha c\theta s\psi c\phi - s\alpha s\theta s\phi + c\alpha s\theta s\psi c\phi - c\alpha c\theta s\phi$$

$$W_{33} = -s\alpha s\psi s\phi c\theta + s\alpha s\theta c\phi + c\alpha s\theta s\phi s\psi + c\alpha c\theta c\phi$$

WHERE $W_{IJ} \left(\begin{matrix} I \\ J \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO WIND MATRIX, WHICH GIVES US:

$$c\gamma = W_{33} = -s\alpha B_{13} + c\alpha B_{33}$$

(ϕ, ψ, θ) (5-12)

$$c\psi_V = W_{22} = s\beta c\alpha B_{12} + c\beta B_{22} + s\beta s\alpha B_{32}$$

WHERE $B_{IJ} \left(\begin{matrix} I \\ J \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

FOR THE GROUND TO BODY SEQUENCE (ψ, ϕ, θ):

$$W_{11} = c\beta c\alpha(c\psi c\theta - s\phi s\psi s\theta) + c\beta s\alpha(s\phi s\psi c\theta + s\theta c\psi) + s\beta c\phi s$$

$$W_{12} = c\beta c\alpha(s\phi s\theta c + s\psi c\theta) + c\beta s\alpha(s\theta s\psi - c\theta c\psi s\phi) - s\beta c\phi c\psi$$

$$W_{13} = -c\beta c\alpha c\phi s\theta + c\beta s\alpha c\theta c\phi - s\beta s\phi$$

$$W_{21} = s\beta c\alpha(c\psi c\theta - s\phi s\psi s\theta) + s\beta s\alpha(c\psi s\theta + s\phi s\psi c\theta) - c\beta c\phi s$$

$$W_{22} = s\beta c\alpha(s\psi c\theta + s\phi c\psi s\theta) + s\beta s\alpha(s\psi s\theta - s\phi c\psi c\theta) + c\beta c\phi s$$

$$W_{23} = -s\beta c\alpha c\phi s\theta + s\beta s\alpha c\phi c\theta + c\beta s\phi$$

$$W_{31} = c\alpha(s\phi s\psi c\theta + c\psi s\theta) + s\alpha(s\phi s\psi s\theta - c\psi c\theta)$$

$$W_{32} = c\alpha(s\psi s\theta - s\phi c\psi c\theta) - s\alpha(s\phi c\psi s\theta + s\psi c\theta)$$

$$W_{33} = c\alpha(c\phi c\theta) + s\alpha c\phi s\theta$$

WHERE $W_{IJ} \left(\begin{matrix} I \\ J \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO WIND MATRIX WHICH GIVES US:

$$c\gamma = W_{33} = -s\alpha B_{13} + c\alpha B_{33}$$

(ψ, ϕ, θ) $\quad c\psi_V = W_{22} = s\beta c\alpha B_{12} + c\beta B_{22} + s\beta s\alpha B_{32}$ (5-13)

WHERE $B_{IJ} \left(\begin{matrix} I \\ J \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

FOR THE GROUND TO BODY SEQUENCE (ψ, θ, ϕ);

$$w_{11} = c\beta\, c\alpha\, c\theta c\psi - s\beta\,(s\phi s\theta c\psi - c\phi s\psi) + c\beta\, s\alpha (c\phi s\theta c\psi + s\phi s\psi)$$

$$w_{12} = c\beta\, c\alpha\, c\theta s\psi - s\beta\,(s\phi s\theta s\psi + c\phi c\psi) + c\beta\, s\alpha (c\phi s\theta\psi - s\phi c\psi)$$

$$w_{13} = -c\beta\, c\alpha\, s\theta - s\beta\, s\phi c\theta + c\beta\, s\alpha\, c\phi c\theta$$

$$w_{21} = s\beta\, c\alpha\, c\theta c\psi + c\beta\,(s\phi s\theta c\psi - c\phi s\psi) + s\beta\, s\alpha (c\phi s\theta c\psi + s\phi s\psi)$$

$$w_{22} = s\beta\, c\alpha\, c\theta s\psi + c\beta (s\phi s\theta\psi + c\phi c\psi) + s\beta\, s\alpha (c\phi c\theta s\psi - s\phi c\psi)$$

$$w_{23} = s\beta\, c\alpha\, s\theta + c\beta\, s\phi c\theta c\psi + s\beta\, s\alpha\, c\phi c\theta$$

$$w_{31} = -s\alpha\, c\theta c\psi + c\alpha\, c\phi s\theta c\psi + c\alpha\, s\phi s\psi$$

$$w_{32} = -s\alpha\, s\theta s\psi + c\alpha (c\phi s\theta s\psi - s\phi c\psi)$$

$$w_{33} = s\alpha\, s\theta + c\alpha\, c\phi c\theta$$

WHERE $w_{ij} \left(\begin{matrix} i \\ j \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO WIND MATRIX WHICH GIVES US:

$$(\psi, \theta, \phi) \quad \begin{aligned} c\gamma &= w_{33} = -s\alpha\, B_{13} + c\alpha\, B_{33} \\ c\psi_v &= w_{22} = s\beta\, c\alpha\, B_{12} + c\beta\, B_{22} + s\beta\, s\alpha\, B_{32} \end{aligned} \tag{5-14}$$

WHERE $B_{ij} \left(\begin{matrix} i \\ j \end{matrix}\right\} = 1, 2, 3\Big)$ IS THE GROUND TO BODY MATRIX FOR THIS SEQUENCE.

TRANSFORMATION COSINE MATRICES FROM KNOWN ORTHOGONAL COMPONENTS OF A VECTOR

A FOURTH METHOD UTILIZED FOR GENERATING THE DIRECTION CONSINES IS BASED ON A KNOWLEDGE OF THE ORTHOGONAL COMPONENTS OF A VECTOR IN THE TWO REFERENCE FRAMES OF INTEREST.

FOR EXAMPLE, IF THE POSITION VECTOR OF POINT M WITH RESPECT TO POINT R (SEE FIGURE 10-1) IS KNOWN IN COMPONENT FORM IN TWO REFERENCE FRAMES $\overline{R}_i$ AND $\overline{R}_1$, WHERE $\overline{S}_i$ IS IN THE DIRECTION OF $\overline{R}_{M/R}$, THEN

$$\overline{R}_{M/R} = x^{R}_{i_{M/R}} \; \overline{R}_i = x^{S}_{M/R} \, \overline{S}_1 \qquad (6\text{-}1)$$

WHICH YIELDS UPON DOTTING WITH $\overline{R}_i$

$$S^{R}_{1_i} = \overline{R}_i \cdot \overline{S}_2 = R^{S}_{i_1} = \sum_{i=1}^{3} \frac{x^{R}_{i_{M/R}} \, \overline{R}_i \, \overline{R}_i}{x^{S}_{M/R}} \qquad (6\text{-}2)$$

ALSO, BY EQ. (6-1)

$$x^{S}_{M/R} = \left[\left(x^{R}_{M/R}\right)^2 + \left(Y^{R}_{M/R}\right)^2 + \left(z^{R}_{M/R}\right)^2\right]^{\frac{1}{2}} \qquad (6\text{-}3)$$

BY EQ. (6-2) IT IS READILY SEEN THAT THREE OF THE DIRECTION COSINES ARE AVAILABLE.

IF THERE ARE ONLY TWO ANGLES INVOLVED, NEITHER OF WHICH ARE ABOUT THE NUMBER-ONE AXIS, ONE MAY HAVE:

$$\overline{S}_i = M_P(\theta) M_Y(\psi) \overline{R}_i,$$

BY FIGURE (3-2)

$$\overline{S}_i = \begin{bmatrix} C\,\theta\, C\,\psi & C\,O\,S\,\psi & -S\,O \\ -S\,\psi & C\,\psi & 0 \\ S\,\theta\, C\,\psi & S\,\theta\, S\,\psi & C\,\theta \end{bmatrix} \overline{R}_i \qquad (6\text{-}4)$$

OR

$$\overline{S}_i = M_Y(\psi) M_P(\theta) \overline{R}_1 \qquad (\psi, \theta)$$

WHICH BY FIGURE (3-5) IS

$$\overline{S}_i = \begin{bmatrix} C\psi\, C\,\theta & S\,\psi & C\,\psi S\,\theta \\ S\psi\, C\,\theta & C\,\psi & S\,\psi S\,\theta \\ S\,\theta & 0 & C\,\theta \end{bmatrix} \overline{F}_i \qquad (6\text{-}5)$$

(θ, ψ)

FOR THE SEQUENCE OF EQ. (6-4 ONE MAY SOLVE FOR THE TWO ANGLES DIVIDING S^R_{12} BY S^R_{11}

$$\frac{S^R_{12}}{S^R_{11}} = \frac{C\,\theta\,S\,\psi}{C\,\theta\,C\,\psi} = \mathrm{TAN}\,\psi_{S/R}\ . \tag{6-6}$$

BY EQ. (6-2) AND (6-6)

$$\frac{S^R_{12}}{S^R_{11}} = \frac{Y^R_{M/R}}{X^F_{M/R}} = \mathrm{TAN}\,\psi_{S/R}\ , \tag{6-7}$$

AND BY EQ. (6-4) AND (6-2)

$$\mathrm{SIN}\,\theta_{S/R} = \frac{Z_{M/R}}{X^S_{M/R}}\ , \tag{6-8}$$

FOR THE SEQUENCE $(\psi,\ \theta)$.
FOR THE SEQUENCE $(\theta,\ \psi)$ BY THE SAME ANALYSIS ONE OBTAINS

$$\mathrm{SIN}\,\psi_{S/R} = \frac{Y^R_{M/R}}{X^S_{M/R}}\ , \tag{6-9}$$

$$-\,\mathrm{TAN}\ \theta_{S/R} = \frac{Z^R_{M/R}}{X^R_{M/R}}\ . \tag{6-10}$$

IN A SIMILAR MANNER, ANGLES OF ATTACK α AND β ARE DETERMINED IN SECTION (13), FROM THE VECTOR EQUATION.

$$\overline{V}_M = V_{M/W}\ \overline{W}_1 = u_{i_{M/W}}\ \overline{B}_i\ . \tag{6-11}$$

AS ANOTHER EXAMPLE, IF THE RECTANGULAR COMPONENTS OF THE MISSILE VELOCITY VECTOR $^R\overline{V}_{M/R} = {}^R\overline{V}_{M/R}\ \ \overline{T}_1 = {}^R\dot{x}_{i_{M/R}}\ \ \overline{R}_i = {}^Ru_{i_{M/R}}\ \ \overline{T}_{i_1}$ (6-12)

WHERE THE $\overline{T}_i$ FRAME IS ORIENTED WITH RESPECT TO THE $\overline{R}_i$ FRAME THROUGH THE SEQUENCE PITCH, YAW, ZERO ROLL,

$$\overline{T}_{i_1} = \begin{bmatrix} C\,\theta_{T/R} & 0 & -\,S\,\theta_{T/R} \\ 0 & 1 & 0 \\ S\,\theta_{T/R} & 0 & C\,\theta_{T/R} \end{bmatrix} \overline{R}_i \tag{6-13}$$

$$\bar{T}_1 = \begin{bmatrix} C\psi_{T/R} & S\psi_{T/R} & 0 \\ -S\psi_{T/R} & C\psi_{T/R} & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{T}_{1_1} \tag{6-14}$$

To solve for the Euler angles, one could replace the components of the vector of Eq. (6-1) and consider the $\bar{S}_1$ frame as the $\bar{T}_1$ frame, hence by Eq. (6-9)

$$\sin\psi_{T/R} = \frac{{}^R\dot{Y}_{M/R}}{{}^RU^S_{M/R}} \tag{6-15}$$

Where

$${}^RU^S_{M/R} = \left[{}^R\dot{X}^2_{M/R} + {}^R\dot{Y}^2_{M/R} + {}^R\dot{Z}^2_{M/R}\right]^{\frac{1}{2}} \tag{6-16}$$

Equation (6-16) requires multiplication and square root operations and would have to be instrumented if this were part of a system design. An alternate expression may be obtained by dotting the second and third term of Eq. (6-12) by $\bar{R}_2$, since observation of the matrix of Figure (3-5) reveals that the term $S_{T/R}$ is equal to $\bar{R}_2 \cdot \bar{T}_1$, therefore

$$\sin\psi_{T/R} = \frac{{}^R\dot{Y}_{M/R}}{{}^RV_{M/R}} \tag{6-17}$$

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Eq. (6-13) still has the term ${}^RV_{M/R}$, but dotting the second and fourth expressions of Eq. (6-12) by $\bar{T}_{11}$

$$\bar{T}_1 \cdot \bar{T}_{11} = C\psi_{T/R} = \frac{{}^RU^{T_1}_{M/R}}{{}^RV_{M/R}} \tag{6-18}$$

dividing Eq. (6-18) into (6-17)

$$\tan\psi_{T/R} = \frac{{}^R\dot{Y}_{M/R}}{{}^RU^{T_1}_{M/R}} \tag{6-19}$$

The demoninator of Eq. (6-19) may be obtained in terms of known quantities ${}^R\dot{X}_{1_{M/R}}$ by dotting the third and fourth terms of Eq. (6-12) by T_{11} and utilizing Eq. (6-17).

$$ {}^{R}U^{T_1}_{M/R} = {}^{R}\dot{X}_{M/R} \; C\,\theta_{T/R} - {}^{R}\dot{Z}_{M/R} \; S\,\theta_{T/R} \quad . \tag{6-20} $$

BY EQ. (6-19) AND (6-20)

$$ \text{TAN}\,\psi_{T/R} = \frac{{}^{R}\dot{Y}_{M/R}}{\dot{X}_{M/R}\, C\,\theta_{T/R} - {}^{R}\dot{Z}_{M/R}\; S\,\theta_{T/R}} \quad , \tag{6-21} $$

AND BY EQ. (6-10)

$$ \text{TAN}\;\theta_{T/R} = \frac{{}^{R}\dot{Z}_{M/R}}{{}^{R}\dot{X}_{M/R}} \quad . \tag{6-22} $$

WHEN DESIGNING A SYSTEM ONE MUST CONSIDER THE TYPE OF COMPUTERS (ANALOG, DIGITAL, ETC.), WHICH WILL BE UTILIZED TO SOLVE THE EQUATIONS, AS WELL AS WHICH SET OF EQUATIONS WILL BE MECHANIZED.

GIMBALED BODY KINEMATICS

THE ANGULAR RELATIONSHIPS BETWEEN REFERENCE FRAMES WHICH ARE PHYSICALLY AVAILABLE AS GIMBAL PICK-OFF ANGLES ARE DISCUSSED IN SECTIONS 7 AND 8. SOME OF THESE ARE STABLE PLATFORMS, FREE GYROS, THREE AXIS FLIGHT TABLES, RADAR ANTENNA.

THREE-GIMBAL STABLE PLATFORM AND PLATFORM ORIENTATION

THE THREE-GIMBAL STABLE PLATFORMS DISCUSSED IN THIS SECTION ARE SHOWN IN FIGURES 7-1. THE PLATFORM PROPER, UPON WHICH THE SENSING ELEMENTS ARE MOUNTED, HAS THREE ANGULAR DEGREES OF FREEDOM WITH RESPECT TO THE MISSILE BODY AXES. IT IS TO BE NOTED THAT THERE ARE TWO DISTINCT GIMBAL "RINGS" (SHOWN AS RECTANGLES), AND THE THIRD GIMBAL IS THE AIRFRAME.

THE TWO DIFFERENT GIMBAL CONFIGURATIONS DISCUSSED ARE: (1) ROLL OUTSIDE GIMBAL, AND (2) PITCH OUTSIDE GIMBAL. THESE ARE THE CASES OF INTEREST WHEN THE PLATFORM "ROTOR" ESTABLISHES A VERTICAL.

SINCE THE PLATFORM FRAME $\overline{F}_i$ IS USED TO ESTABLISH AN AIRBORNE REFERENCE FRAME, THE MISSILE BODY FRAME $\overline{B}_i$ WILL BE ORIENTED WITH RESPECT TO THE P_i FRAME. THE TRANSFORMATION MATRIX AS A FUNCTION OF GIMBAL PICKOFF ANGLES IS OBTAINED AS FOLLOWS:

CONSIDER THE $\overline{B}_i$ FRAME INITIALLY ALIGNED WITH THE $\overline{P}_i$ FRAME AND A GIMBAL CONFIGURATION FROM INSIDE OUT: YAW GIMBAL, ROLL GIMBAL, AND PITCH GIMBALL AS SHOWN IN FIGURE 7-1.

THE FIRST ROTATION (FROM INSIDE TO OUTSIDE) IS A YAW ABOUT THE $\overline{P}_i$ AXIS. THIS ROTATION CAN BE MEASURED, AS SHOWN IN FIGURE 7-2, BY MEASURING THE ANGLE G_s, POSITIVE FOR A POSITIVE ROTATION, AS SHOWN. THE DIRECTION COSINE MATRIX FOR THIS ROTATION IS GIVEN BY EQ. (2.1) (WHEN THE YAW ANGLE SYMBOL IS CHANGED FROM ψ TO G_s) AS

$$\begin{pmatrix} \overline{B}_{11} \\ \overline{B}_{21} \\ \overline{B}_{31} \end{pmatrix} = \begin{pmatrix} \cos G_s & \sin G_s & 0 \\ -\sin G_s & \cos G_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{P}_1 \\ \overline{P}_2 \\ \overline{P}_3 \end{pmatrix} \qquad (7\text{-}1)$$

$$\overline{B}_{i1} = M_Y(G_s)\ \overline{P}_i$$

THE SECOND ROTATION IS A ROLL THROUGH AN ANGLE DESIGNATED BY G_i WHICH CAN BE MEASURED BETWEEN THE INNER AND OUTER GIMBAL RINGS.

The roll matrix is given by Eq. (2-3) as

$$\begin{pmatrix} \bar{B}_{12} \\ \bar{B}_{23} \\ \bar{B}_{32} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos G_I & \sin G_I \\ 0 & -\sin G_I & \cos G_I \end{pmatrix} \begin{pmatrix} \bar{B}_{11} \\ \bar{B}_{21} \\ \bar{B}_{31} \end{pmatrix} \tag{7-2}$$

$$\bar{B}_{i2} = M_R(G_I)\bar{B}_{i1}$$

The final rotation about the body-fixed axis is a pitch. The matrix is given by Eq. (2.2)

$$\begin{pmatrix} \bar{B}_{1} \\ \bar{B}_{2} \\ \bar{B}_{3} \end{pmatrix} = \begin{pmatrix} \cos G_o & 0 & -\sin G_o \\ 0 & 1 & 0 \\ \sin G_o & 0 & \cos G_o \end{pmatrix} \begin{pmatrix} \bar{B}_{12} \\ \bar{B}_{22} \\ \bar{B}_{32} \end{pmatrix} \tag{7-3}$$

$$\bar{B}_i = M_P(G_o)\bar{B}_{i2}$$

By Eqs. (7-1), (7-2), and (7-3), the transformation matrix is

$$\bar{B}_i = M_P(G_o) M_R(G_I) M_Y(G_S)\bar{P}_i \tag{7-4}$$

Performing the indicated matrix multiplication of Eq. (7-4), one obtains the matrix for a pitch-outside gimbal platform.

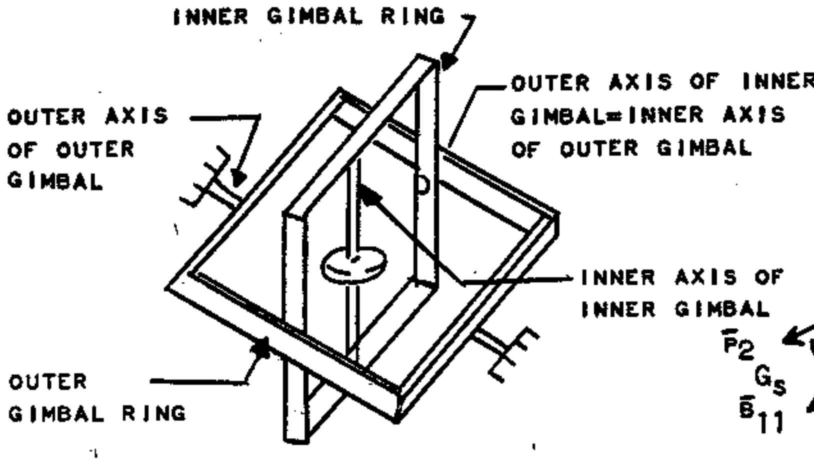
$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \cos G_o \cos G_S - \sin G_o \sin G_o \sin G_I & \cos G_o \sin G_S + \cos G_S \sin G_o \sin G_I & -sG_o cG_I \\ -\sin G_S \cos G_I & \cos G_I \cos G_S & \sin G_I \\ \sin G_o \cos G_S + \sin G_S \sin G_I \cos G_o & \sin G_o \sin G_S - \cos G_S \sin G_I \cos G_o & cG_o cG_I \end{pmatrix} \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{pmatrix} \tag{7-5}$$

The orientation of the $\bar{B}_i$ frame with respect to the $\bar{P}_i$ frame through the sequence of Euler angles ψ, ϕ, Θ is given by Figure (3-1) as

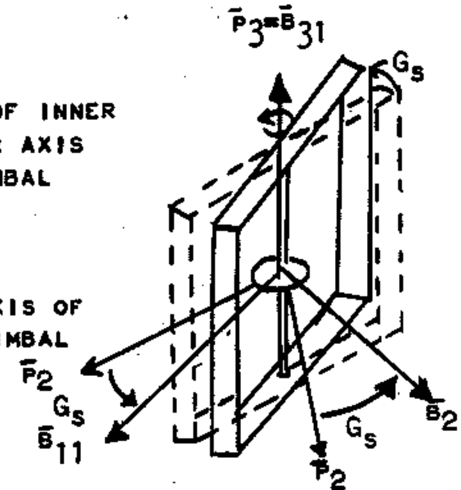
$$\bar{B}_i = M_P(\Theta)\, M_R(\phi)\, M_Y(\psi)\, \bar{P}_i \tag{7-6}$$

It is obvious that Eqs. (7-6) and (7-4) are identical when the angle symbols are equated as follows:

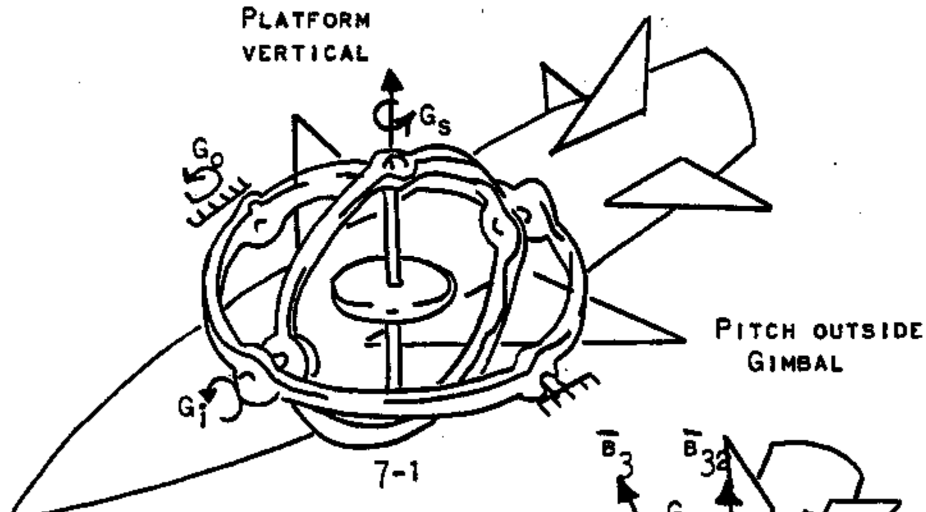
$$\begin{aligned} \psi &= G_S \\ \phi &= G_I \\ \Theta &= G_o \end{aligned} \tag{7-7}$$


INNER GIMBAL RING
OUTER AXIS OF INNER GIMBAL=INNER AXIS OF OUTER GIMBAL
OUTER AXIS OF OUTER GIMBAL
INNER AXIS OF INNER GIMBAL
OUTER GIMBAL RING

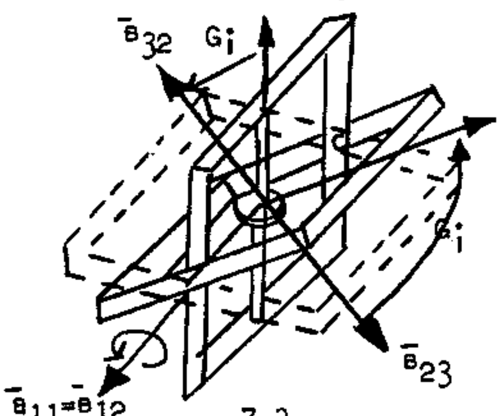
7-5


$\bar{P}_3=\bar{B}_{31}$
G_s
$\bar{P}_2$
G_s
$\bar{B}_{11}$
G_s
$\bar{B}_{21}$
$\bar{P}_2$

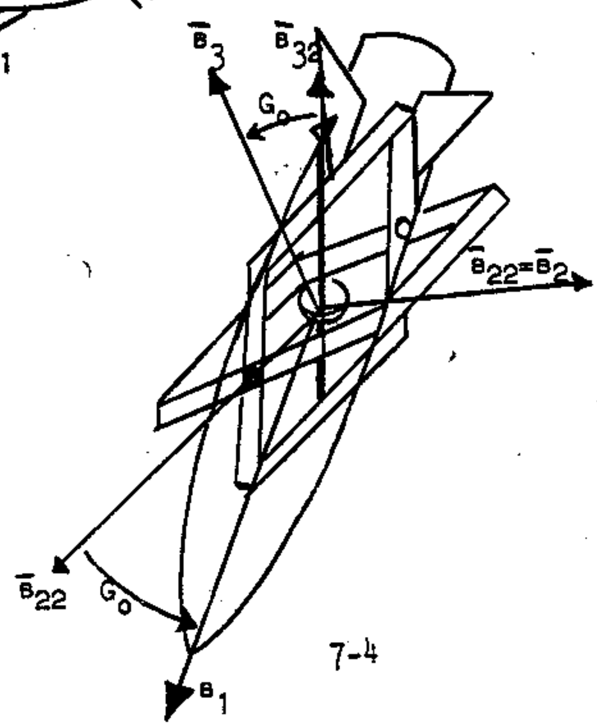
7-2


PLATFORM VERTICAL
G_s
G_o
G_i
PITCH OUTSIDE GIMBAL

7-1


$\bar{B}_{31}=\bar{P}_3$
$\bar{B}_{32}$
G_i
G_i
$\bar{B}_{23}$
$\bar{B}_{11}=\bar{B}_{12}$

7-3


$\bar{B}_3$
$\bar{B}_{32}$
G_o
$\bar{B}_{22}=\bar{B}_2$
$\bar{B}_{22}$
G_o
$\bar{B}_1$

7-4

The relationships of Eq. (7-7) are to be expected, since the gimbal pick-off angles fulfill the requirements of successive rotations. The first gimbal angle is about an axis of the platform frame; the last is about one of the missile, axes, while the second is about an axis mutually perpendicular to the first and third. Furthermore, G_s is always about the $\bar{P}_3$ axis, G_o is always about an axis fixed in the airframe; G_i is about an axis which is mutually perpendicular to $\bar{P}_3$; and the third rotation axis is fixed in the airframe ($\bar{B}_2$ in this case). It is to be observed that each gimbal ring has rigidly fixed to it two orthogonal axes of rotation, an inner axis of rotation, and an outer axis of rotation. Thus, as shown in Figure (7-5) the outer axis of the inner gimbal ring is the inner axis of the outer gimbal ring.

Consider a platform having the gimbal configuration: yaw gimbal inside, pitch gimbal, roll gimbal outside, as shown in Figure (7-1-b). Again, obtaining the orientation of $\bar{B}_i$ with respect to $\bar{P}_i$, from inside gimbal angles to outside, one obtains the matrix equation,

$$\bar{B}_i = M_R(G_o)\, M_P(G_i) M_Y(G_s) \bar{P}_i \tag{7-8}$$

Using the roll, pitch, and yaw matrices of Eqs. (7-1), (7-2), and (7-3) the following matrix is obtained for a roll outside gimbal platform.

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} cG_i\, cG_s & cG_i\, sG_s & -sG_i \\ sG_o\, sG_i\, cG_s - cG_o\, sG_s & sG_s\, sG_i\, sG_o + cG_o\, cG_s & sG_o\, cG_i \\ cG_o\, sG_i\, cG_s + sG_o\, sG_s & cG_o\, sG_i\, sG_s - sG_o\, cG_s & cG_o\, cG_i \end{pmatrix} \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{pmatrix} \tag{7-9}$$

As before, setting $P_i = B_i$ and $F_i = P_i$, the matrix of Figure (3-2) can be equated to Eq. (7-8). Using the property of equality of the elements of equal matrices, the Euler angles and gimbal pickoff angles for the Euler sequence yaw, pitch, roll, and roll outside gimbal platform are:

$$\begin{aligned} \psi &= G_s \\ \theta &= G_i \\ \phi &= G_o \end{aligned} \tag{7-10}$$

From the above consideration and Eqs. (7-7) and (7-10) the general statement can be made that:

For a given gimbal configuration, there exists a unique Euler angle sequence such that there will be a one-to-one correspondence between the gimbal pickoff angles and the Euler angles of that sequence.

Thus, the direction cosines required for coordinate transformation can be computed directly from the gimbal pickoff angles.

GYROS

THE ORIENTATION OF THE MISSILE WITH RESPECT TO THE LOCAL HORIZONTAL FRAME CAN BE OBTAINED BY MEANS OF GYROS. VARIOUS COMBINATIONS OF TWO TWO-DEGREES-OF-FREEDOM GYROS (THE SPIN AXIS OF ONE SLAVED TO THE LOCAL VERTICAL, AND THE SPIN AXIS OF THE OTHER SLAVED NORMAL TO THE LOCAL VERTICAL) WILL BE CONSIDERED IN THIS REPORT. THAT IS, COMBINATIONS OF A VERTICAL GYRO AND A DIRECTIONAL GYRO WILL BE CONSIDERED.

THE DIFFERENTIAL EQUATIONS OF (3.26) RELATE THE ANGULAR RATES OF THE REFERENCE FRAMES, THE EULER ANGLES, AND THE EULER ANGULAR RATES. WHEN USING THESE DIFFERENTIAL EQUATIONS TO DETERMINE THE ANGULAR RATES OF ONE REFERENCE FRAME KNOWLEDGE OF THE COMPONENTS OF THE ANGULAR RATE OF THE OTHER REFERENCE FRAME AND THE SET OF EULER ANGLES, OR THE SET OF EULER ANGULAR RATES (BUT NOT NECESSARILY BOTH OF THE LATTER SETS, SINCE THE ANGLE CAN BE OBTAINED FROM ITS RATE BY AN INTEGRATION) IS REQUIRED. WHEN USING THE DIFFERENTIAL EQUATIONS TO DETERMINE THE EULER ANGLES, THEN BOTH SETS OF THE COMPONENTS OF THE ANGULAR RATES OF THE TWO REFERENCE FRAMES MUST BE KNOWN. A RESUME OF SEVERAL GYRO COMBINATIONS USED FOR THE ESTABLISHMENT OF AN ANGULAR REFERENCE FRAME IS GIVEN IN REFERENCE 1.

CONSIDER A SLAVED TWO-DEGREE-OF-FREEDOM GYRO, SUCH AS SHOWN IN FIGURE (8-1). EVEN THOUGH THE ROTOR HAS THREE ANGULAR DEGREES OF FREEDOM WITH RESPECT TO THE GYRO CASE, THE PLANE OF THE INNER GIMBAL RING, WHICH CONTAINS THE ANGULAR MOMENTUM VECTOR OF THE ROTOR, HAS ONLY TWO ANGULAR DEGREES OF FREEDOM WITH RESPECT TO THE CASE. THE TWO $\bar{P}_i$ VECTORS IN THE PLANE OF THE PLATFORM LOSE SIGNIFICANCE WHEN THE PLATFORM IS CONSIDERED AS A GYRO, FOR THESE VECTORS NOW LIE IN THE PLANE OF THE GYRO WHEEL WHICH IS SPINNING AT A HIGH RATE. CONSEQUENTLY, THE REMAINING AXIS, WHICH IS ALONG THE ROTOR SPIN AXIS, IS THE SIGNIFICANT AXIS. THE TWO ANGLES WHICH SPECIFY ITS ORIENTATION ARE, OF COURSE, G_o AND G_i. THE THIRD ANGLE, G_s, IS THE TIME INTEGRAL OF THE ROTOR SPIN RATE AND IS NOT USED.

ATTITUDE ANGLES USING AN EAST-SLAVED GYRO AND A VERTICAL SLAVED GYRO WITH PITCH-OUTSIDE GIMBAL

THE ATTITUDE ANGLES OF THE MISSILE CAN BE OBTAINED BY TWO SLAVED TWO-DEGREE-OF-FREEDOM GYROS IN THE FOLLOWING MANNER. CONSIDER THE GYRO GIMBAL CONFIGURATIONS AS SHOWN IN FIGURE (8-3) HAVING A PITCH OUTSIDE GIMBAL.

THE TRANSFORMATION MATRIX IS THE SAME AS FOR A STABLE PLATFORM AND IS GIVEN BY EQ. (7-5) AS

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} [cG_{ov}cG_{sv} - sG_{sv}G_{ov}sG_{iv}] & [cG_{ov}sG_{sv} + cG_{sv}sG_{ov}sG_{iv}] & [-sG_{ov}cG_{iv}] \\ -sG_{sv}cG_{iv} & cG_{iv}cG_{sv} & sG_{iv} \\ [sG_{ov}cG_{sv} + sG_{sv}sG_{iv}cG_{ov}] & [sG_{ov}sG_{sv} - cG_{sv}sG_{iv}cG_{ov}] & [cG_{ov}cG_{iv}] \end{pmatrix} \begin{pmatrix} \bar{P}_{1v} \\ \bar{P}_{2v} \\ \bar{P}_{3v} \end{pmatrix} \tag{8-1}$$

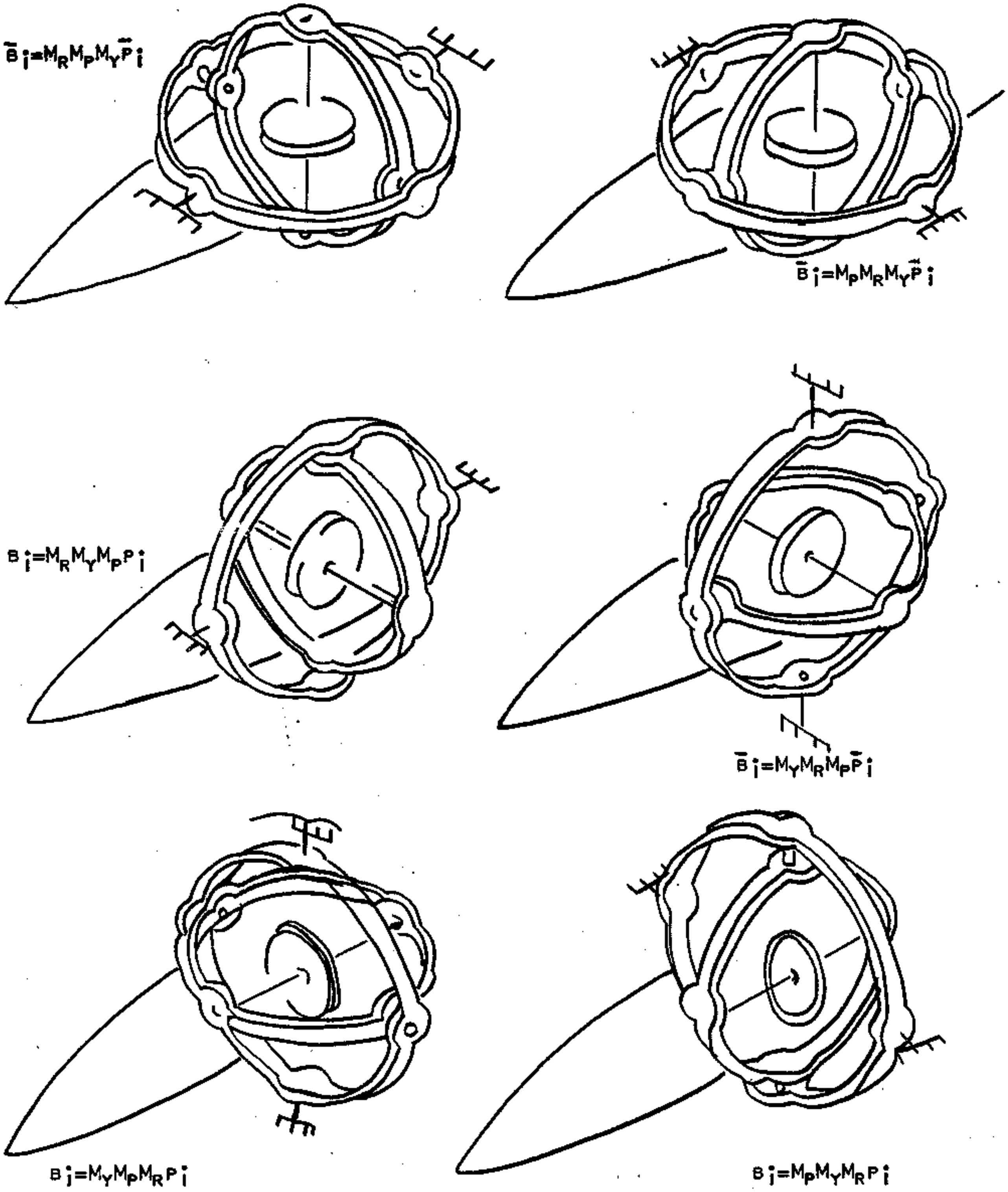


FIG. (8-1). TWO DEGREE OF FREEDOM GYRO CONFIGURATIONS.

FOR A VERTICAL GYRO, THE THIRD COLUMN OF THIS MATRIX EQUATION HAS SIGNIFICANCE. THAT IS

$$\bar{P}_{3v} \equiv \bar{G}_3 \tag{8.2}$$

AND HENCE BY EQS. (8.1) AND (8.2),

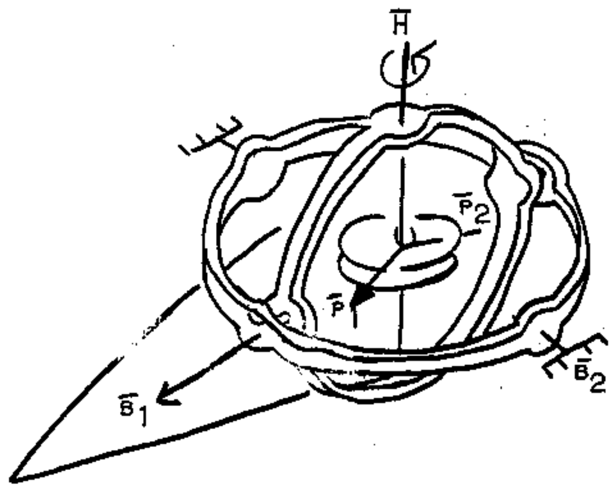
$$\bar{B}_1 \cdot \bar{G}_3 = sG_{ov}cG_{Iv} = B_{13} \tag{8.3}$$

$$\bar{B}_2 \cdot \bar{G}_3 = sG_{Iv} = B_{23} \tag{8.4}$$

$$\bar{B}_3 \cdot \bar{G}_3 = cG_{ov}cG_{Iv} = B_{33} \tag{8.5}$$

THE ORIENTATION OF THE B_i FRAME WITH RESPECT TO THE LOCAL HORIZONTAL FRAME USING THE EULER SEQUENCE ψ, ϕ, θ IS GIVEN WITH FIGURE (3-1) AS

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} c\theta c\psi - s\psi s\theta s\phi & c\theta s\psi + c\psi s\theta s\phi & -s\theta c\phi \\ -s\psi c\phi & c\phi c\psi & s\phi \\ s\theta c\psi + s\psi s\phi c\theta & s\theta s\psi - c\psi s\phi c\theta & c\theta c\phi \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix} \tag{8.6}$$



$$\bar{H} = (Iw_s)\bar{P}_{3v} = (Iw_s)\bar{G}_3$$

FIG. (8-2) TWO-DEGREE OF FREEDOM VERTICAL GYRO WITH PITCH GIMBAL OUTSIDE.

BY EQ. (8.6)

$$\bar{B}_1 \cdot \bar{G}_1 = -s\theta c\phi \tag{8.7}$$

$$\bar{B}_2 \cdot \bar{G}_2 = s\phi \tag{8.8}$$

$$\bar{B}_3 \cdot \bar{G}_3 = c\theta c\phi \tag{8.9}$$

TWO OF THE EULER ANGLES ARE THE GIMBAL PICKOFF ANGLES, FOR BY EQS. (8.4-8.8) AND (8.3-8.7)

$$G_{iv} = \phi \tag{8.10}$$

$$G_{ov} = \theta \tag{8.11}$$

IT IS TO BE OBSERVED THAT TO FULFILL THE INITIAL ASSUMPTIONS IN DERIVING THE DIRECTION COSINE MATRICES FOR THE SIX EULER SEQUENCES THAT THE $\bar{B}_i$ FRAME SHOULD BE ORIGINALLY ALIGNED WITH THE $\bar{P}_i$ FRAME, I.E. $\bar{B}_i = \bar{P}_{10} = \bar{B}_2 = \bar{P}_{20}$, AND $\bar{B}_3 = \bar{P}_{30}$. HOWEVER, FOR CONVENIENCE, THE $\bar{P}_i$ AXES ARE CHANGED AROUND WHEN CONSIDERING THE HORIZONTAL GYROS, WITH THE ROTOR AXIS TAKING ON THE SUBSCRIPT CORRESPONDING TO THE $\bar{B}_i$ SUBSCRIPT OF THE INITIAL ORIENTATION. BY ADOPTING THIS CONVENTION, THE PREVIOUSLY DERIVED TRANSFORMATION MATRICES CAN BE UTILIZED.

CONSIDER A DIRECTIONAL GYRO AS SHOWN IN FIGURE (8-3). THE TRANSFORMATION MATRIX IS FIRST A PITCH,

$$\bar{B}_{i1} = M_P(G_{sH})\, \bar{P}_{iH} \tag{8.12}$$

SECOND A ROLL, $\bar{B}_{i2} = M_R(G_{iH})\, \bar{B}_{i1}$ (8.13)

AND THIRD A YAW, $\bar{B}_i = M_Y(G_{oH})\, \bar{B}_{i2}$ (8.14)

BY EQS. (8.12), (8.13) AND (8.14), ONE OBTAINS

$$\bar{B}_i = M_Y(G_{oH})M_R(G_{iH})M_P(G_{sH})\bar{P}_{iH} \tag{8.15}$$

EQ. (8.15) CORRESPONDS TO THE EULER SEQUENCE θ, ϕ, ψ AND BY FIGURE 8-1

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} cG_{oH}cG_{sH} + sG_{oH}\, sG_{iH}\, sG_{sH} & [sG_{oH}cG_{iH}] & -cG_{oH}sG_{sH} + sG_{oH}sG_{iH}cG_{sH} \\ -sG_{oH}cG_{sH} + cG_{oH}\, sG_{iH}\, sG_{sH} & [cG_{oH}cG_{iH}] & sG_{oH}sG_{sH} + cG_{oH}sG_{iH}cG_{sH} \\ cG_{iH}sG_{sH} & [-sG_{iH}] & cG_{iH}cG_{sH} \end{pmatrix} \begin{pmatrix} \bar{P}_{1H} \\ \bar{P}_{2H} \\ \bar{P}_{3H} \end{pmatrix} \tag{8.17}$$

This gyro is erected such that the spin axis indicates the local north direction, therefore,

$$\bar{F}_{2H} = \bar{G}_2 \tag{8.17}$$

by Eqs. (8.16) and (8.17),

$$\bar{B}_1 \cdot \bar{G}_2 = sG_{oH}oG_{iH} = B_{12} \tag{8.18}$$

$$\bar{B}_2 \cdot \bar{G}_2 = cG_{oH}cG_{iH} = B_{22} \tag{8.19}$$

$$\bar{B}_3 \cdot \bar{G}_2 = sG_{iH} = B_{32} \tag{8.20}$$

The direction cosine matrix between the body frame and local horizontal frame is given as

$$\begin{array}{c|ccc} & \bar{G}_1 & \bar{G}_2 & \bar{G}_3 \\ \hline \bar{B}_1 & B_{11} & B_{12} & B_{13} \\ \bar{B}_2 & B_{21} & B_{22} & B_{23} \\ \bar{B}_3 & B_{31} & B_{32} & B_{33} \end{array} \tag{8.21}$$

The second column of Eq. (8.21) is given by Eqs. (8.18), (8.19) and (8.20); and the third column of Eq. (8.21) is given by Eqs. (8.3), (8.4) and (8.5).

The direction cosines of Eq. (8.21) are given by Eq. (8.16) for the Euler sequence ψ, ϕ, θ. Using the property of the cosine matrix that each element is equal to its cofactor, as developed in Eq. (1.13), the first column of Eq. (8.21) is given by Eqs. (1-13) as,

$$\begin{aligned} B_{11} &= B_{22}B_{33} - B_{23}B_{32} \\ B_{21} &= B_{13}B_{32} - B_{12}B_{33} \\ B_{31} &= B_{12}B_{23} - B_{13}B_{22} \end{aligned} \tag{8.22}$$

Expressing the first column of Eq. (8.21) in terms of the vertical and horizontal gimbal pickoff angles given by Eqs. (8.3), (8.4), (8.5) and (8.13), (8.19), (8.20),

$$\begin{aligned} B_{11} &= (cG_{oH}cG_{iH})\,(cG_{ov}cG_{iv}) + sG_{iv}sG_{iH} \\ B_{21} &= sG_{ov}cG_{iv}sG_{iH} - sG_{oH}cG_{iH}cG_{ov}cG_{iv} \\ B_{31} &= (sG_{oH}cG_{iH})sG_{iv} + sG_{ov}cG_{iv}cG_{oH}cG_{iH} \end{aligned} \tag{8.23}$$

Figure (3-6) shows this gyro pair and the corresponding matrix.

Equating matrices of Eq. (8-6)(for the ψ, ϕ, θ sequence) the matrix of Figure 8-3, and the Euler matrix with Figure 3-2 (for the $\bar{\psi}$, $\bar{\theta}$, $\bar{\phi}$ sequence)

$$= \begin{pmatrix} c\theta c\psi - s\psi s\theta s\phi & c\theta s\psi + c\psi s\theta s\phi & -s\theta c\phi \\ -s\psi c\phi & c\phi c\psi & s\phi \\ s\theta c\psi + s\psi s\phi c\theta & s\theta s\psi - c\psi s\phi c\theta & c\theta c\phi \end{pmatrix}_{\psi,\ \phi,\ \theta}$$

$$= \begin{pmatrix} cG_{oH}cG_{iH}cG_{oV}cG_{iV} + sG_{iV}sG_{iH} & sG_{oH}cG_{iH} & -sG_{oV}cG_{iV} \\ sG_{oV}cG_{iV}sG_{iH} - s\,G_{oH}cG_{iH}cG_{oV}c\,G_{iV} & cG_{oH}cG_{iH} & sG_{iV} \\ sG_{oH}cG_{iH}sG_{iV} + sG_{oV}cG_{iV}cG_{oH}cG_{iH} & -sG_{oH} & cG_{oV}cG_{iV} \end{pmatrix} \tag{8.24}$$

$$= \begin{pmatrix} c\bar{\theta}\,c\bar{\psi} & c\bar{\theta}\,s\bar{\psi} & -s\bar{\theta} \\ s\bar{\phi}s\bar{\theta}\,c\bar{\psi} - c\bar{\phi}s\bar{\psi} & s\bar{\psi}\,s\bar{\theta}\,s\bar{\phi} + c\bar{\phi}c\bar{\psi} & s\bar{\phi}c\bar{\theta} \\ c\bar{\phi}s\bar{\theta}\,c\bar{\psi} + s\bar{\phi}s\bar{\psi} & c\bar{\phi}s\bar{\theta}\,s\bar{\psi} - s\bar{\phi}c\bar{\psi} & c\bar{\phi}c\bar{\theta} \end{pmatrix}_{\bar{\psi},\ \bar{\theta},\ \bar{\phi}}.$$

Equating elements of the second row, second column

$$c\phi c\psi = c\,G_{oH}cG_{iH}$$

$$c\,\psi = \frac{cG_{oH}cG_{iH}}{cG_{iV}} \tag{8.25}$$

since $\phi = G_{iV}$.

Solving (8.25) for ψ and rewriting (8-10) and (8-11)

$$\psi = \text{arc c}\ \frac{cG_{oH}cG_{iH}}{cG_{iV}} \tag{8.26}$$

$$\phi = G_{iV}$$

$$\theta = G_{oV}$$

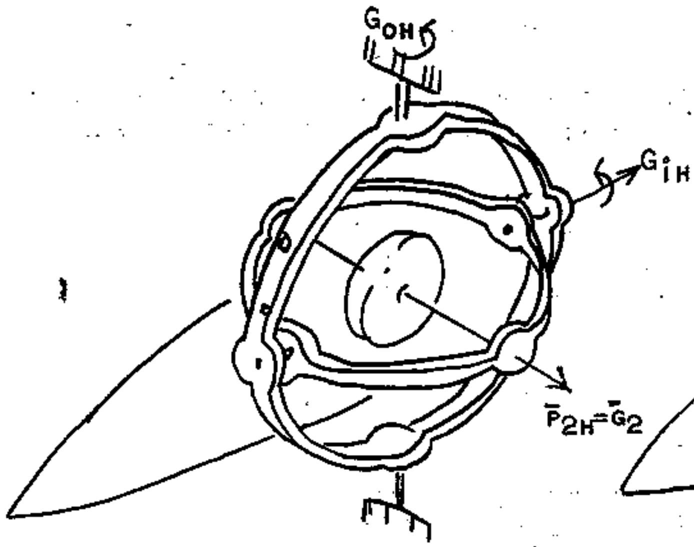
Thus the Euler angles ψ, ϕ, and θ can be obtained using the pickoff angles, G_{iV}, G_{oV}, G_{oH} and G_{iH}.

The Euler angles for the sequence $\bar{\psi}$, $\bar{\theta}$, $\bar{\phi}$ are obtained from the latter two equalities of Eq. (8.24). Equating elements of the third column, one obtains

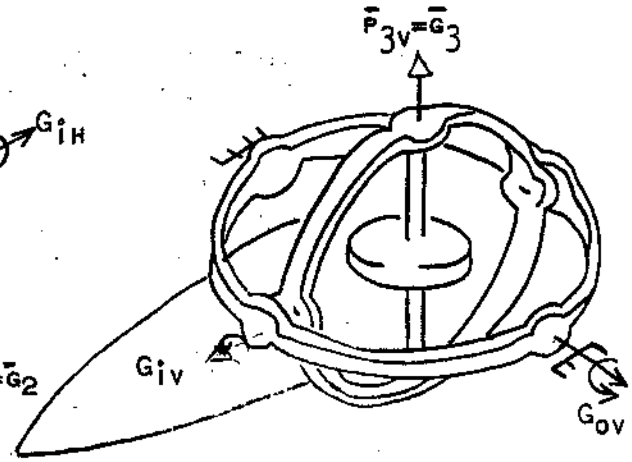
$$-s\bar{\theta} = -\,sG_{oV}cG_{iV} \tag{8.27}$$

$$s\bar{\phi}\,c\bar{\theta} = s\,G_{iV} \tag{8.28}$$

$$c\bar{\phi}\,c\bar{\theta} = cG_{oV}cG_{iV} \tag{8.29}$$



DIRECTIONAL (NORTH) GYRO
YAW GIMBAL OUTSIDE



PITCH OUTSIDE VERTICAL
GYRO

$$\psi = c^{-1}\left[\frac{cG_{oH}cG_{iH}}{cG_{iv}}\right]$$

$$\phi = G_{iv}$$

$$\theta = G_{ov}$$

FOR EULER SEQUENCE (, ϕ, θ)

	G_1	G_2	G_3
$\bar{B}_1$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{iv}sG_{iH}$	$sG_{oH}cG_{iH}$	$-sG_{oH}cG_{iv}$
$\bar{B}_2$	$sG_{ov}cG_{iv}sG_{iH} - sG_{oH}cG_{iH}cG_{ov}cG_{iv}$	$sG_{oH}cG_{iH}$	sG_{iv}
$\bar{B}_3$	$sG_{oH}cG_{iH}sG_{iv} + sG_{ov}cG_{iv}cG_{oH}cG_{iH}$	$-sG_{oH}$	$cG_{ov}cG_{iv}$

FIG. (8-3). GYROS FOR VEHICLE REQUIRING 90° PITCH.

From Eq. (8.27)

$$\Theta = \text{arc s}\left[sG_{ov}cG_{iv}\right]. \tag{8.30}$$

Dividing Eq. (8.28) by (8.27),

$$\tan\bar{\Phi} = \tan G_{iv}\sec G_{ov} \tag{8.31}$$

$$\bar{\Phi} = \text{arc tan}\left[\tan G_{iv}\sec G_{ov}\right]$$

Dividing the element of the first row, second column, by the element of the first row, first column, and equating it to the corresponding element ratio of gimbal angle matrix, the following is obtained,

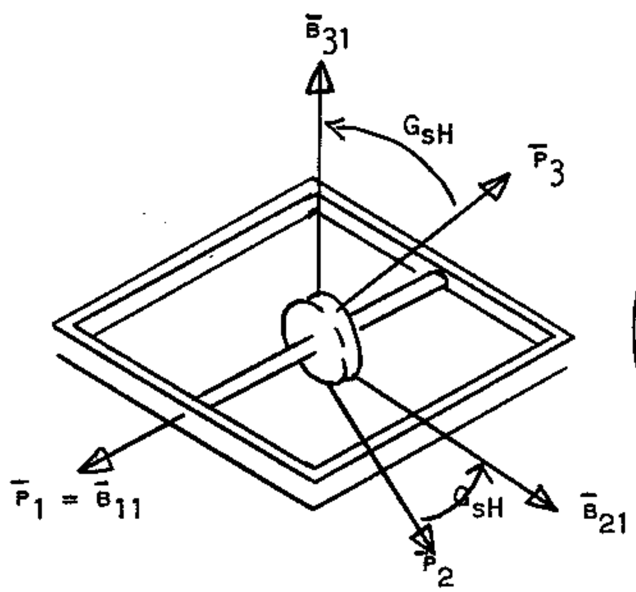
$$\tan\bar{\Psi} = \frac{sG_{oH}cG_{iH}}{cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{iv}sG_{iH}} \tag{8.32}$$

$$\tan\bar{\Psi} = \frac{1}{\tan G_{oH}cG_{ov}cG_{iv} + sG_{iv}\csc G_{oH}\tan G_{iH}}$$

The heading angle $\bar{\Psi}$ can be obtained from Eq. (8-32)

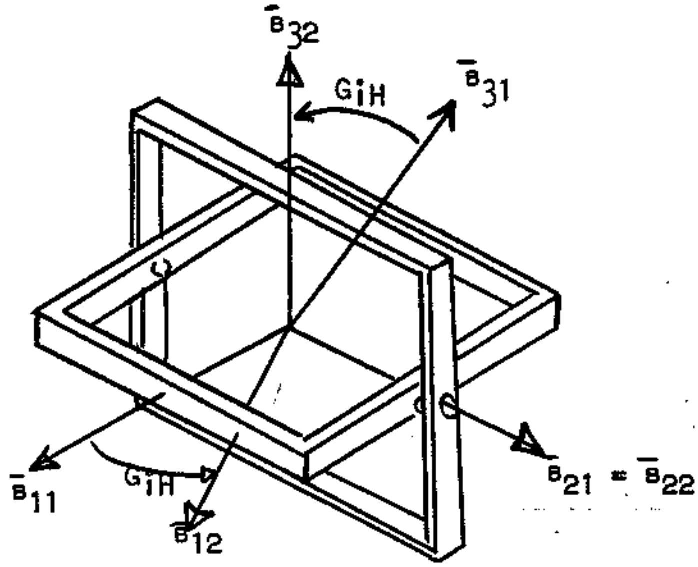
Attitude Angles Using East-Slaved and Vertical-Slaved Gyros

Consider the gyro combination as given in Figure (8-4). The positive rotations are as shown:



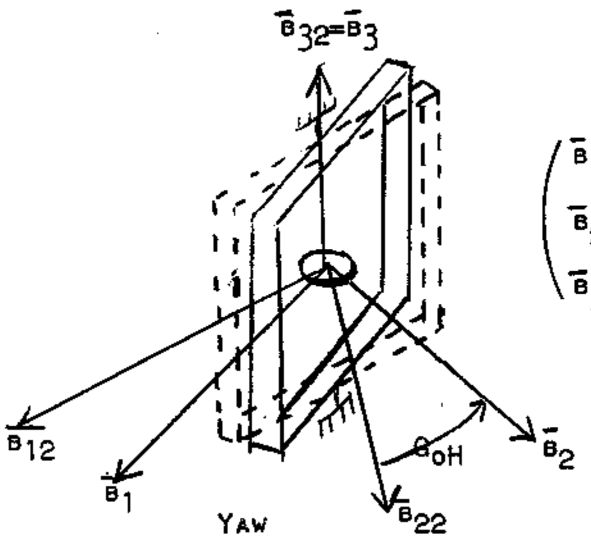
$$\begin{pmatrix} \bar{B}_{11} \\ \bar{B}_{12} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cG_{sH} & sG_{sH} \\ 0 & -sG_{sH} & sG_{sH} \end{pmatrix} \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{pmatrix} \tag{8.33}$$

Fig. 8.3a. Body Axis After First Rotation.



$$\begin{pmatrix} \bar{B}_{12} \\ \bar{B}_{22} \\ \bar{B}_{32} \end{pmatrix} = \begin{pmatrix} cG_{iH} & 0 & -sG_{iH} \\ 0 & 1 & 0 \\ sG_{iH} & 0 & cG_{iH} \end{pmatrix} \begin{pmatrix} \bar{B}_{11} \\ \bar{B}_{21} \\ \bar{B}_{31} \end{pmatrix} \tag{8.34}$$

FIG. 8.3B. BODY AXIS AFTER SECOND ROTATION.



$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} cG_{oH} & sG_{oH} & 0 \\ -sG_{oH} & cG_{oH} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{B}_{12} \\ \bar{B}_{22} \\ \bar{B}_{32} \end{pmatrix} \tag{8-35}$$

FIG. 8.3C. BODY AXIS AFTER THIRD ROTATION.

By Eqs. (8-33), (3-34), and (8-35),

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} cG_{oH}cG_{iH} & sG_{oH}cG_{sH} + sG_{sH}cG_{oH} & sG_{iH}sG_{sH}sG_{oH}sG_{sH} \\ -sG_{oH}cG_{iH} & cG_{oH}cG_{sH} - sG_{oH}sG_{iH}sG_{sH} - cG_{oH}sG_{iH}cG_{sH} & \\ sG_{iH} & -cG_{iH}sG_{sH} & cG_{iH}cG_{sH} \end{pmatrix} \begin{pmatrix} \bar{P}_{1H} \\ \bar{P}_{2H} \\ \bar{P}_{3H} \end{pmatrix} \qquad (8\text{-}36)$$

Since the above gyro is slaved to the $\bar{G}_1$ (local east vector) (8-37)

$$\bar{G}_1 = \bar{P}_{1H}$$

By Eqs. (8.36) and (8.37)

$$\begin{aligned} \bar{B}_1 \cdot \bar{G}_1 &= cG_{oH}cG_{iH} = B_{11} \\ \bar{B}_2 \cdot \bar{G}_1 &= -sG_{oH}cG_{iH} = B_{21} \\ \bar{B}_3 \cdot \bar{G}_1 &= sG_{iH} = B_{31} \end{aligned} \qquad (8\text{-}38)$$

Thus for the gyro pair given by Figure (8-4) the first column of the direction cosine matrix is given by Eq. (8-38), the third column given by Eq. (8-3), (8-4), and (8-5), as before, and the second column by the following relations obtained from Eq. (1-13)

$$\begin{aligned} B_{12} &= B_{23}B_{31} - B_{21}B_{33} \\ B_{22} &= B_{11}B_{33} - B_{13}B_{31} \\ B_{32} &= B_{13}B_{21} - B_{11}B_{23} \end{aligned} \qquad (8\text{-}39)$$

The gyro pair given by Figure 8-3 can pitch 90° without gimbal lock; however, if it rolls 90°, gimbal, lock will occur in both gyros.

The gyro pair given by Figure 8-4 will have gimbal lock on the vertical gyro if it rolls 90°, or gimbal lock on the directional gyro if it pitches 90°.

The attitude angles for the gyro pair of Figure 8-4 can be obtained by the following equality,

$$\begin{pmatrix} c\theta c\psi - s\psi s\theta s\phi & c\theta s\psi + c\psi s\theta s\phi & -s\theta c\phi \\ -s\psi c\phi & c\phi c\psi & s\phi \\ s\theta c\psi + s\psi s\phi c\theta & s\theta s\psi - c\psi s\phi c\theta & c\theta c\phi \end{pmatrix} = \quad \psi,\ \phi,\ \theta$$

$$
\begin{pmatrix}
cG_{oH}cG_{iH} & sG_{iv}sG_{iH} + sG_{oH}cG_{iH}cG_{ov}cG_{iv} & -sG_{ov}cG_{iv} \\
-sG_{oH}cG_{iH} & cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{iH} & sG_{iv} \\
sG_{iH} & sG_{ov}cG_{iv}sG_{oH}cG_{iH} - cG_{oH}cG_{iH}sG_{iv} & sG_{ov}cG_{iv}
\end{pmatrix}
= \tag{8-40}
$$

$$
\begin{pmatrix}
c\Theta c\Psi & c\Theta s\Psi & -s\Theta \\
s\Phi s\Theta c\Psi - c\Phi s\Psi & s\Psi s\Theta s\Phi + c\Phi c\Psi & s\Phi c\Theta \\
c\Phi s\Theta c\Psi + s\Phi s\Psi & c\Phi s\Theta s\Psi - s\Phi c\Psi & c\Phi c\Theta
\end{pmatrix}_{\Psi,\Theta,\Phi}
$$

THE EULER ANGLES FOR THE SEQUENCE ψ, Φ, θ ARE:

$$\phi = G_{iv} \tag{8-41}$$

$$\theta = G_{ov} \tag{8-42}$$

AND

$$c\,\psi = \frac{cG_{oH}cG_{iH}sG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{iH}}{\cos G_{iv}}$$

$$c\,\psi = cG_{oH}cG_{iH}sG_{ov} + sG_{ov}sG_{iH} \tag{8-43}$$

THE EULER ANGLES FOR THE SEQUENCE Ψ, Φ, Θ ARE OBTAINED FROM THE SECOND TWO EQUALITIES OF EQ. (8-40) AND ARE

$$\Theta = \text{ARC S } sG_{ov}cG_{iv} \tag{8-44}$$

$$\Phi = \text{ARC TAN } \tan G_{iv} \sec G_{ov} \tag{8-45}$$

THE HEADING ANGLES Ψ IS

$$\tan \Psi = \frac{sG_{iv}sG_{iH} + sG_{oH}cG_{iH}cG_{ov}cG_{iv}}{cG_{oH}cG_{iH}} \tag{8-46}$$

<u>ATTITUDE ANGLES USING A HORIZONTAL GYRO AND ROLL OUTSIDE VERTICAL GYRO</u>

CONSIDER A VERTICAL GYRO WITH THE GIMBAL CONFIGURATION OF FIGURE 8-5. THIS GYRO HAS A ROLL OUTSIDE GIMBAL. THE TRANSFORMATION MATRIX IS GIVEN BY EQ. (7-9) BY ADDING THE SUBSCRIPT "V" AND SETTINGS $P_3 = G_3$

$$
\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix}
=
\begin{pmatrix}
cG_{iv}cG_{sv} & cG_{iv}sG_{sv} & -sG_{iv} \\
sG_{ov}sG_{iv}sG_{sv} - cG_{ov}sG_{sv} & sG_{sv}sG_{iv}sG_{ov} + cG_{ov}cG_{sv} & sG_{ov}cG_{iv} \\
cG_{ov}sG_{iv}cG_{sv} + sG_{ov}sG_{sv} & cG_{ov}sG_{iv}sG_{sv} - sG_{ov}cG_{sv} & cG_{ov}cG_{iv}
\end{pmatrix}
\begin{pmatrix} \bar{P}_{1v} \\ \bar{P}_{2v} \\ \bar{J}_3 \end{pmatrix}
\tag{8-47}
$$

By Figure 3-2 the third column as a function of the Euler angles are equated to column three of Eq. (8-47)

$$
\begin{aligned}
\bar{B}_1 \cdot \bar{P}_3 &= B_{13} = -sG_{iv} &&= s\Theta \\
\bar{B}_2 \cdot \bar{P}_3 &= B_{23} = sG_{ov}cG_{iv} &&= s\Phi c\Theta \\
\bar{B}_3 \cdot \bar{P}_3 &= B_{33} = cG_{ov}cG_{iv} &&= c\Phi c\Theta
\end{aligned}
\tag{8-48}
$$

Hence for the Euler sequence Ψ, Θ, Φ, the roll angle Φ and the elevation angle Θ is

$$
\begin{aligned}
\Theta &= G_{iv} \\
\Phi &= G_{ov}
\end{aligned}
$$

If the horizontal gyro of Figure 3-2 is used, the second column of the transformation matrix is

$$
\begin{aligned}
\bar{B}_1 \cdot \bar{G}_2 &= sG_{oH}cG_{iH} = B_{12} \\
\bar{B}_2 \cdot \bar{G}_2 &= cG_{oH}cG_{iH} = B_{22} \\
\bar{B}_3 \cdot \bar{G}_2 &= -sG_{oH} \quad = B_{32}
\end{aligned}
\tag{8-50}
$$

The first column of the direction cosine matrix is given by Eq. 1-13 as

$$
\begin{aligned}
B_{11} &= B_{22}B_{33} - B_{23}B_{32} \\
B_{21} &= B_{13}B_{32} - B_{12}B_{33} \\
B_{31} &= B_{12}B_{23} - B_{13}B_{22}
\end{aligned}
\tag{8-51}
$$

Thus, by the relations (8-48), (8-50), and (8-51), the matrix of Figure 8-5 is obtained. The attitude angles are obtained as before by replacing the gimbal pickoff matrix of Eq. (8-40) with the gimbal matrix of Figure (3-4)

$$
\begin{pmatrix}
c\theta c\psi - s\psi s\theta s\phi & c\theta s\psi + c\psi s\theta s\phi & -s\theta c\phi \\
-s\psi c\phi & c\phi c\psi & s\phi \\
s\theta c\psi + s\psi s\phi c\theta & s\theta s\psi - c\psi s\phi c\theta & c\theta c\phi
\end{pmatrix}_{\psi,\ \phi,\ \theta} =
$$

$$
\begin{pmatrix}
cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{oH} & sG_{oH}cG_{iH} & -sG_{iv} \\
sG_{iv}G_{oH} - sG_{oH}cG_{iH}cG_{ov}cG_{iv} & cG_{oH}cG_{iH} & sG_{ov}cG_{iv} \\
sG_{oH}cG_{iH}sG_{ov}cG_{iv} + sG_{iv}cG_{oH}cG_{iH} & -sG_{oH} & cG_{ov}cG_{iv}
\end{pmatrix} = \tag{8-52}
$$

$$
\begin{pmatrix}
c\cancel{\theta} c\psi & c\cancel{\theta} s\psi & -s\cancel{\theta} \\
s\underline{\phi} s\theta c\psi - c\underline{\phi} s\psi & s\psi s\theta s\underline{\phi} + c\underline{\phi} c\psi & s\underline{\phi} c\cancel{\theta} \\
c\underline{\phi} s\cancel{\theta} c\psi + s\underline{\phi} s\psi & c\underline{\phi} s\cancel{\theta} s\psi - s\underline{\phi} c\psi & c\underline{\phi} c\cancel{\theta}
\end{pmatrix}_{\psi, \theta, \phi}
$$

THE ATTITUDE ANGLES FOR THE SEQUENCE $\psi, \cancel{\theta}, \underline{\phi}$ ARE GIVEN BY EQ. 8-49. THE HEADING ANGLE IS OBTAINED FOR EQ. 8-51 AS

$$\tan \psi = \frac{sG_{oH}cG_{iH}}{cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{oH}}$$

OR

$$\tan \psi = \frac{1}{\tan G_{oH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}\sec G_{iH}} \tag{8-53}$$

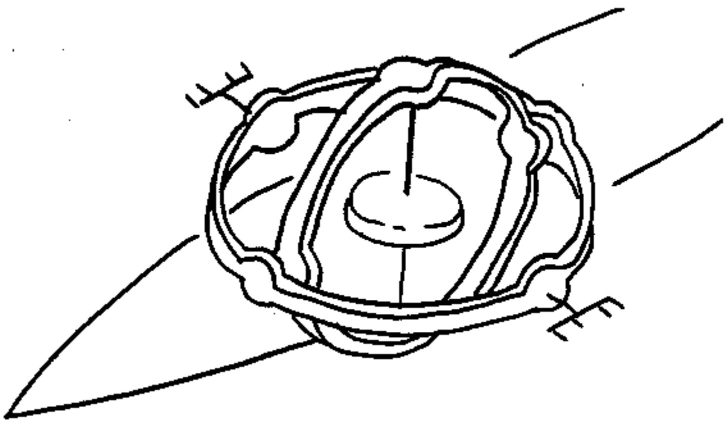
THE EULER ANGLES FOR THE SEQUENCE ψ, ϕ, θ ARE OBTAINED FROM THE LEFT EQUALITIES OF EQ. (8-51).

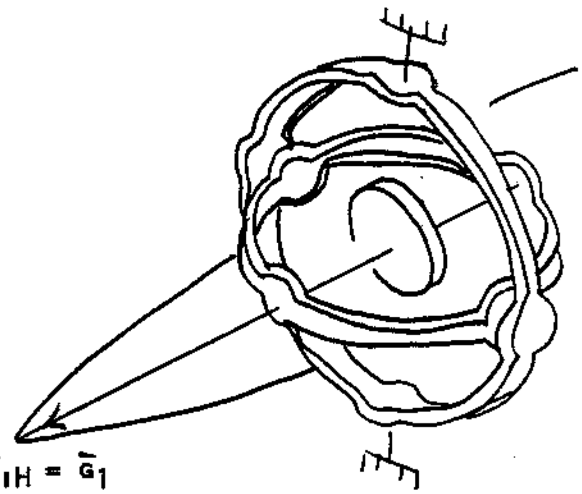
$$\phi = \text{arc s}\,(sG_{ov}cG_{iv}) \tag{8-54}$$

$$\theta = \text{arc tan}\,(\tan G_{iv}\sec G_{ov}) \tag{8-55}$$

$$\tan\psi = \frac{sG_{iv}sG_{oH} - sG_{oH}cG_{iH}cG_{ov}cG_{iv}}{-cG_{oH}cG_{iH}} \tag{8-56}$$

THUS, VARIOUS OTHER COMBINATIONS OF VERTICAL AND DIRECTIONAL GYROS CAN SIMILARLY BE ANALYZED.





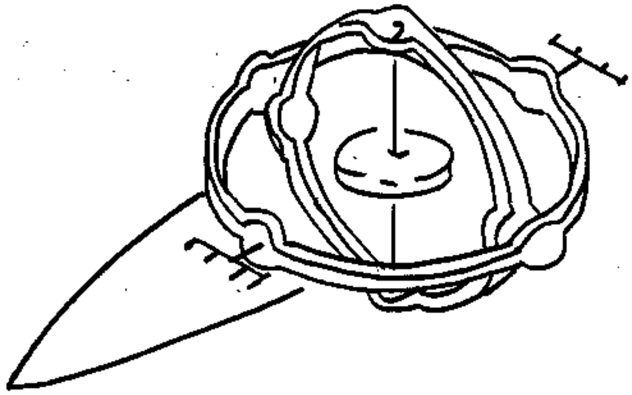
	G_1	G_2	G_3
B_1	$cG_{oH}cG_{iH}$	$sG_{iv}sG_{iH} + sG_{oH}cG_{iH}cG_{ov}cG_{iv}$	$-sG_{ov}cG_{iv}$
B_2	$-sG_{oH}cG_{iH}$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{iH}$	sG_{iv}
B_3	sG_{iH}	$sG_{ov}cG_{iv}sG_{oH}cG_{iH} - cG_{oH}cG_{iH}sG_{iv}$	$cG_{ov}cG_{iv}$

$$G_{iv} = \phi$$

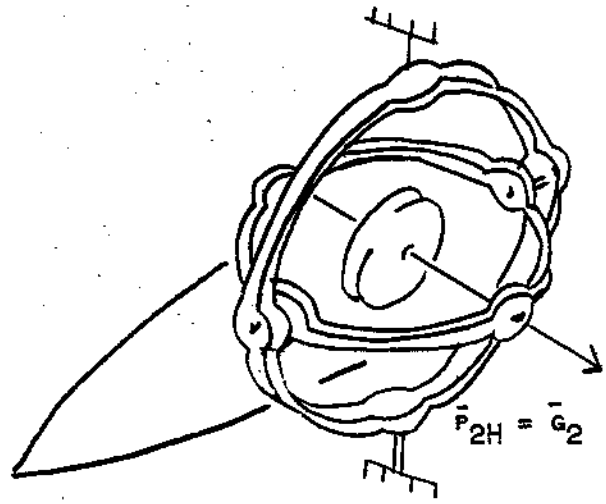
$$G_{ov} = \Theta$$

FOR THE EULER SEQUENCE ψ, ϕ, Θ

FIG. (8-4). TWO DEGREE OF FREEDOM GYRO PAIR.



ROLL OUTSIDE GIMBAL VERTICAL GYRO



DIRECTIONAL (NORTH) GYRO, YAW GIMBAL OUTSIDE

	G_1	G_2	G_3
$\bar{B}_1$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{oH}$	$sG_{oH}sG_{iH}$	$-sG_{iv}$
$\bar{B}_2$	$sG_{iv}sG_{oH} - sG_{oH}cG_{iH}cG_{ov}cG_{iv}$	$cG_{oH}cG_{iv}$	$sG_{ov}cG_{iv}$
$\bar{B}_3$	$sG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{iv}sG_{oH}cG_{iH}$	$-sG_{oH}$	$cG_{ov}cG_{iv}$

FIG. (8-5). DIRECTION COSINE MATRIX FROM VERTICAL AND DIRECTIONAL (NORTH) GYROS.

$\overline{B}_i = M_R\, M_Y\, M_Y\, \overline{P}_i$

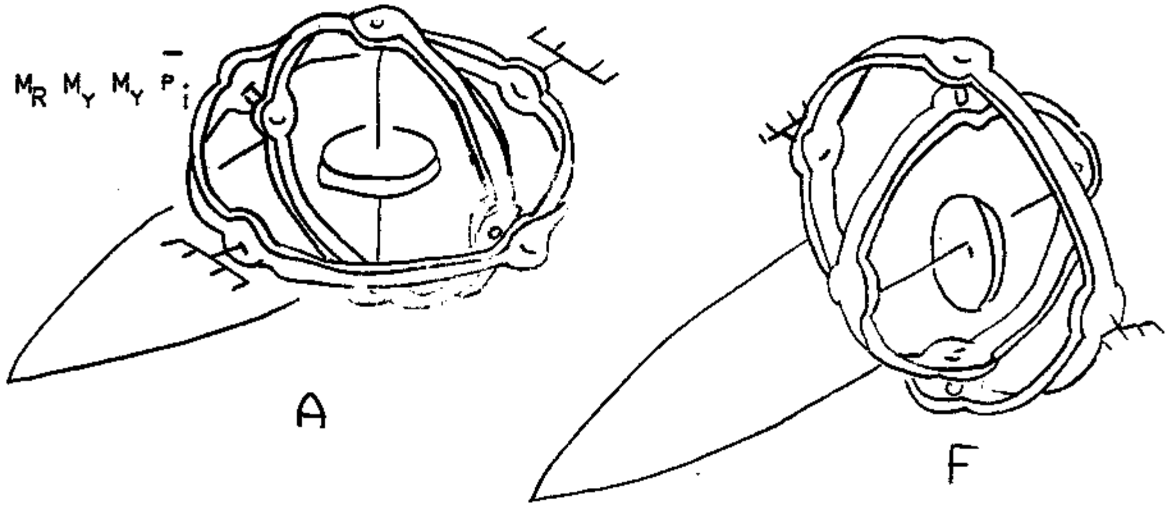


FIG. 8-1-A $\overline{B}_i = M_R\, M_P\, M_Y\, \overline{P}_i$ AND FIG. 8-1-F $\overline{B}_i = M_P\, M_Y\, M_R\, \overline{P}_i$

$$\left.\begin{aligned} G_{iV} &= \theta \\ G_{oV} &= \phi \end{aligned}\right\} \text{FOR SEQUENCE } (\Psi,\ \theta,\ \phi) \qquad \left.\begin{aligned} G_{iH} &= \Psi \\ G_{oH} &= \theta \end{aligned}\right\} \text{FOR SEQUENCE } (\phi, \Psi, \theta)$$

$$\Psi = \cos^{-1} cG_{oH}\, cG_{iH}\, \sec G_{iV}$$

$$|\theta| = G_{iV}$$

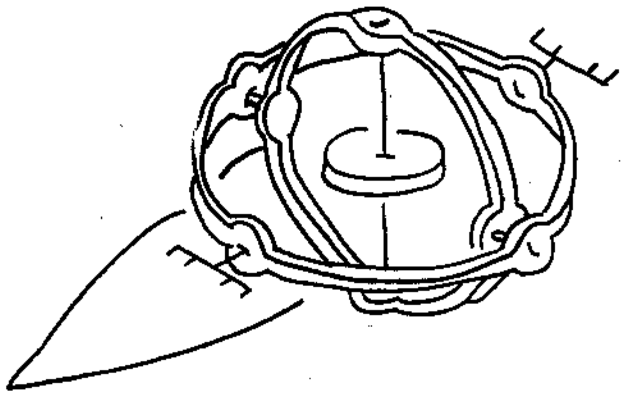
$$\phi = G_{oV}$$

	$\overline{G}_1$	$\overline{G}_2$	$\overline{G}_3$
$\overline{B}_1$	$cG_{oH}cG_{iH}$	$-sG_{iH}cG_{oV}cG_{iV} - sG_{oV}cG_{iV}sG_{oH}cG_{iH}$	$-sG_{iV}$
$\overline{B}_2$	$-sG_{iH}$	$cG_{oH}cG_{oV}cG_{iV} + sG_{iV}sG_{oH}cG_{iH}$	$sG_{oV}cG_{iV}$
$\overline{B}_3$	$sG_{oH}cG_{iH}$	$cG_{oH}cG_{iH}sG_{oV}cG_{iV} - sG_{iV}sG_{iH}$	$cG_{oV}cG_{iV}$

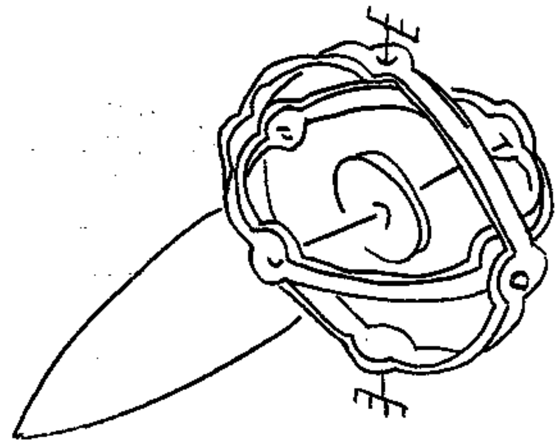
GIMBAL LOCK WHEN $G_{iV} = 90°$

$G_{iH} = 90°$

FIG. 8-6.



A



E

FIGURE 8-1-A ($\bar{B}_i = M_R M_P M_Y \bar{P}_i$) AND FIGURE 8-1-E ($\bar{B}_i = M_Y M_P M_R \bar{P}_i$)

$$\left.\begin{array}{l} G_{iV} = \theta \\ G_{oV} = \phi \end{array}\right\} \text{FOR SEQUENCE } (\psi, \theta, \phi) \qquad \left.\begin{array}{l} G_{iH} = \theta \\ G_{oH} = \psi \end{array}\right\} \text{FOR SEQUENCE } (\phi, \theta, \psi)$$

$$\Psi = \cos^{-1}(cG_{iH}cG_{oH}\sec G_{iV})$$

$$\Theta = G_{iV}$$

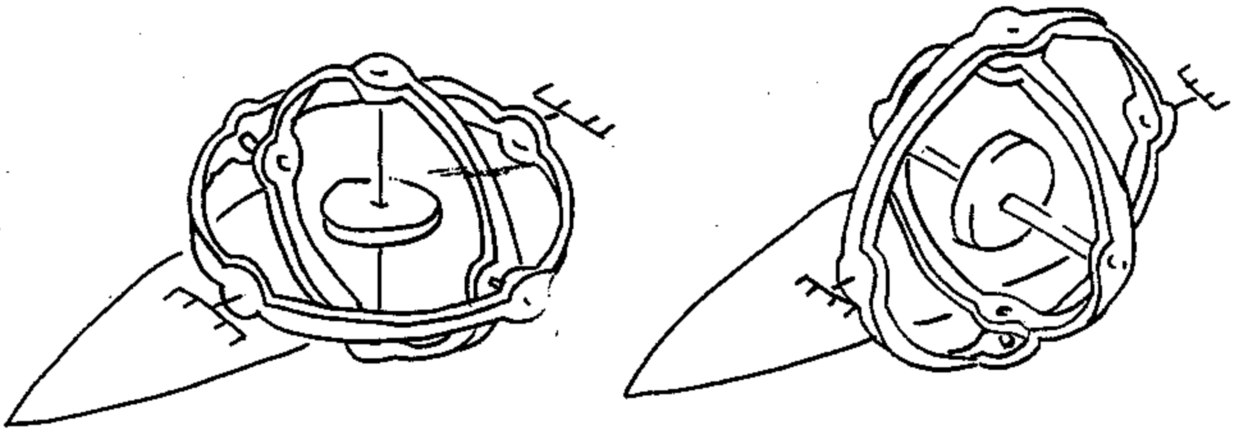
$$\Phi = G_{oV}$$

	$\bar{G}_1$	$\bar{G}_2$	$\bar{G}_3$
$\bar{B}_1$	$cG_{iH}cG_{oH}$	$-cG_{iH}sG_{oH}cG_{oV}cG_{iV} - sG_{oV}cG_{iV}sG_{iH}$	$-sG_{iV}$
$\bar{B}_2$	$-cG_{iH}sG_{oH}$	$cG_{iH}cG_{oH}cG_{oV}cG_{iV} + sG_{iV}sG_{iH}$	$sG_{oV}cG_{iV}$
$\bar{B}_3$	sG_{iH}	$cG_{iH}cG_{oH}sG_{oV}cG_{iV} - sG_{iV}cG_{iH}sG_{oH}$	$cG_{oV}cG_{iV}$

GIMBAL LOCK WHEN $G_{iV} = 90°$

$G_{iH} = 90°$

FIG. 8-7.



A C

FIGURE 8-1-A ($\bar{B}_i = M_R M_P M_Y \bar{P}_i$) AND FIGURE 8-1-C ($\bar{B}_i = M_R M_Y M_P \bar{P}_i$

$G_{iv} = \theta$, $G_{ov} = \phi$ $\Big\}$ FOR SEQUENCE (ψ, θ, ϕ) $G_{iH} = \psi$, $G_{oH} = \phi$ $\Big\}$ FOR SEQUENCE (θ, ψ, ϕ)

$$\bar{\Psi} = \sin^{-1}(sG_{iH} \sec G_{iv})$$

$$\bar{\Theta} = G_{iv}$$

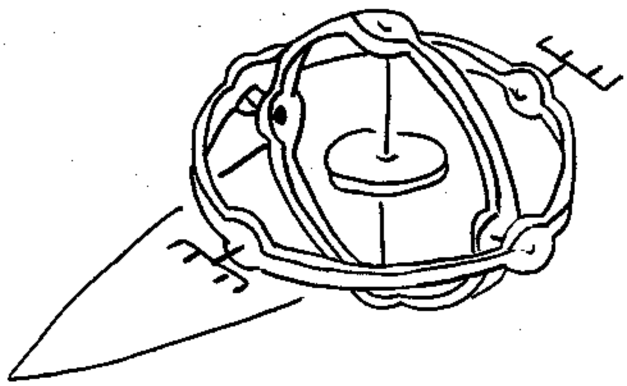
$$\bar{\Phi} = G_{ov}$$

	$\bar{G}_1$	$\bar{G}_2$	$\bar{G}_3$
$\bar{B}_1$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{oH}cG_{iH}$	sG_{iH}	$-sG_{iv}$
$\bar{B}_2$	$sG_{iH}cG_{ov}cG_{iv} - sG_{iv}sG_{oH}cG_{iH}$	$cG_{oH}cG_{iH}$	$-sG_{ov}cG_{iv}$
$\bar{B}_3$	$sG_{iH}sG_{ov}cG_{iv} + sG_{iv}cG_{oH}cG_{iH}$	$-sG_{oH}cG_{iH}$	$cG_{ov}cG_{iv}$

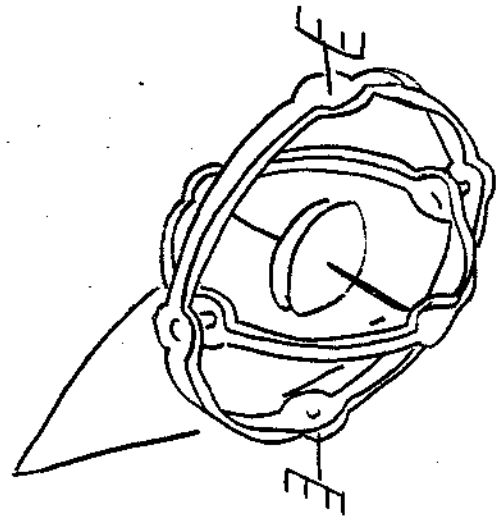
GIMBAL LOCK WHEN $G_{iv} = 90°$

$G_{iH} = 90°$

FIG. 8-8.



A



D

FIGURE 8-1-A ($\bar{B}_i = M_R M_P M_Y \bar{P}_i$) AND FIGURE 8-1-D ($\bar{B}_i = M_Y M_R M_P \bar{P}_i$)

$\left.\begin{array}{l} G_{iv} = \theta \\ G_{ov} = \phi \end{array}\right\}$ FOR SEQUENCE (ψ, θ, ϕ)

$\left.\begin{array}{l} G_{iH} = \phi \\ G_{oH} = \psi \end{array}\right\}$ FOR SEQUENCE (θ, ϕ, ψ)

$$\bar{\psi} = \sin^{-1}(sG_{oH}cG_{iH}\sec G_{iv})$$

$$\bar{\Theta} = G_{iv}$$

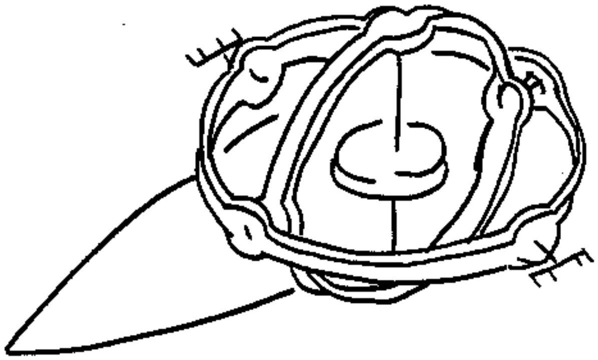
$$\bar{\Phi} = G_{ov}$$

	$\bar{G}_1$	$\bar{G}_2$	$\bar{G}_3$
$\bar{B}_1$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{iH}$	$sG_{oH}cG_{iH}$	$-sG_{iv}$
$\bar{B}_2$	$sG_{oH}cG_{iH}cG_{ov}cG_{iv} - sG_{iv}sG_{iH}$	$cG_{oH}cG_{iH}$	$sG_{ov}cG_{iv}$
$\bar{B}_3$	$sG_{oH}cG_{iH}sG_{ov}cG_{iv} + sG_{iv}cG_{oH}cG_{iH}$	$-sG_{iH}$	$cG_{ov}cG_{iv}$

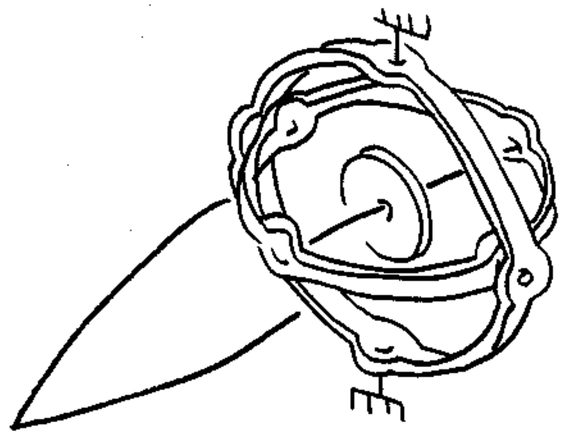
GIMBAL LOCK WHEN $G_{iv} = 90°$

$G_{iH} = 90°$

FIG. 8-9.



B



E

FIGURE 8-1-B ($\bar{B}_i = M_P M_R M_Y \bar{P}_i$) AND FIGURE 8-1-E ($\bar{B}_i = M_Y M_P M_R \bar{P}_i$)

$$\left.\begin{aligned} G_{iv} &= \phi \\ G_{ov} &= \theta \end{aligned}\right\} \text{FOR SEQUENCE } (\psi, \phi, \theta) \qquad \left.\begin{aligned} G_{iH} &= \theta \\ G_{oH} &= \psi \end{aligned}\right\} \text{FOR SEQUENCE } (\phi, \theta, \psi)$$

$$\bar{\Psi} = \tan^{-1}\left[-\tan G_{oH} cG_{ov} cG_{iv} - (\tan G_{iH} sG_{iv}) \sec G_{oH}\right]$$

$$\Theta = \sin^{-1}(sG_{ov} cG_{iv})$$

$$\bar{\Phi} = \tan^{-1}(\tan G_{iv} \sec G_{ov})$$

	$\bar{G}_1$	$\bar{G}_2$	$\bar{G}_3$
$\bar{B}_1$	$cG_{iH}cG_{oH}$	$-cG_{iH}sG_{oH}cG_{ov}cG_{iv} - sG_{iv}sG_{iH}$	$-sG_{ov}cG_{iv}$
$\bar{B}_2$	$-cG_{iH}sG_{oH}$	$cG_{iH}cG_{oH}cG_{ov}cG_{iv} + sG_{ov}cG_{iv}sG_{iH}$	sG_{iv}
$\bar{B}_3$	sG_{iH}	$cG_{iH}cG_{oH}sG_{iv} - sG_{ov}cG_{iv}cG_{iH}sG_{oH}$	$cG_{ov}cG_{iv}$

GIMBAL LOCK WHEN $G_{iv} = 90°$

$G_{iH} = 90°$

FIG. 8-10.

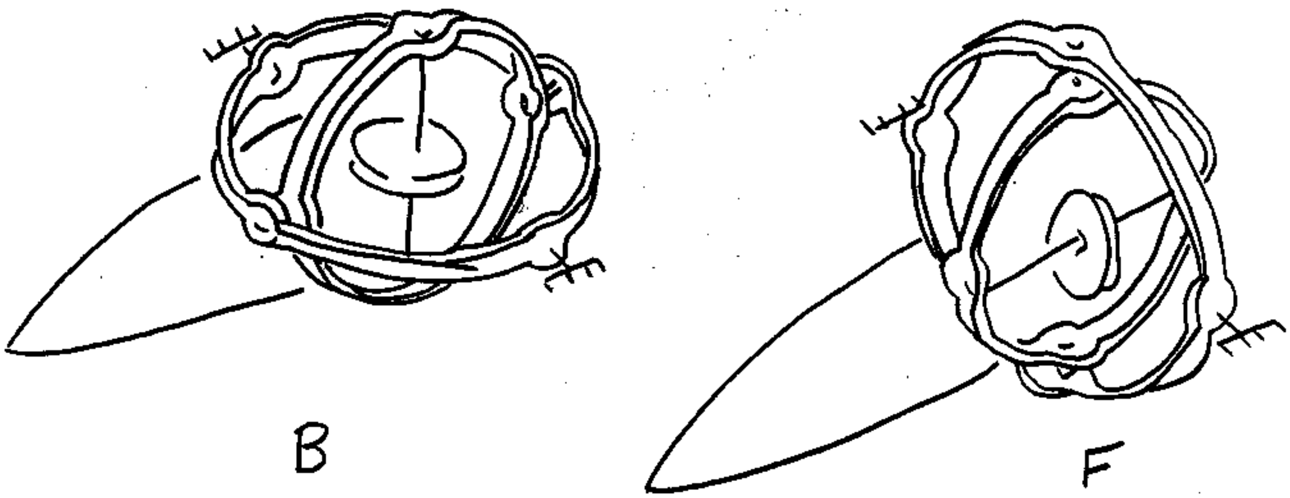


FIGURE 8-1-B ($\overline{B}_i = M_P M_R M_Y \overline{P}_i$) AND FIGURE 8-1-F ($\overline{B}_i = M_P M_Y M_R \overline{P}_i$)

$$\left.\begin{aligned} G_{iV} &= \phi \\ G_{oV} &= \theta \end{aligned}\right\} \text{FOR SEQUENCE } (\psi, \phi, \theta) \qquad \left.\begin{aligned} G_{iH} &= \psi \\ G_{oH} &= \theta \end{aligned}\right\} \text{FOR SEQUENCE } (\phi, \psi, \theta)$$

$$\Psi = \tan^{-1}\left[(-\tan G_{iH}\, cG_{oV}\, cG_{iV})\sec G_{oH} - \tan G_{oH}\, sG_{iV}\right]$$

$$\Theta = \sin^{-1}(sG_{oV}\, cG_{iV})$$

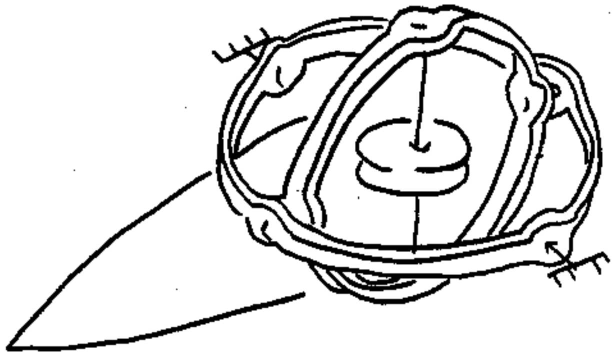
$$\Phi = \tan^{-1}(\tan G_{iV} \sec G_{oV})$$

	$\overline{G}_1$	$\overline{G}_2$	$\overline{G}_3$
$\overline{B}_1$	$cG_{oH}cG_{iH}$	$-sG_{iH}cG_{oV}cG_{iV} - sG_{oH}cG_{iH}sG_{iV}$	$-sG_{oV}cG_{iV}$
$\overline{B}_2$	$-sG_{iH}$	$cG_{oH}cG_{iH}cG_{oV}cG_{iV} + sG_{oV}cG_{iV}sG_{oH}cG_{iH}$	sG_{iV}
$\overline{B}_3$	$sG_{oH}cG_{iH}$	$cG_{oH}cG_{iH}sG_{iV} - sG_{oV}cG_{iV}sG_{iH}$	$cG_{oV}cG_{iV}$

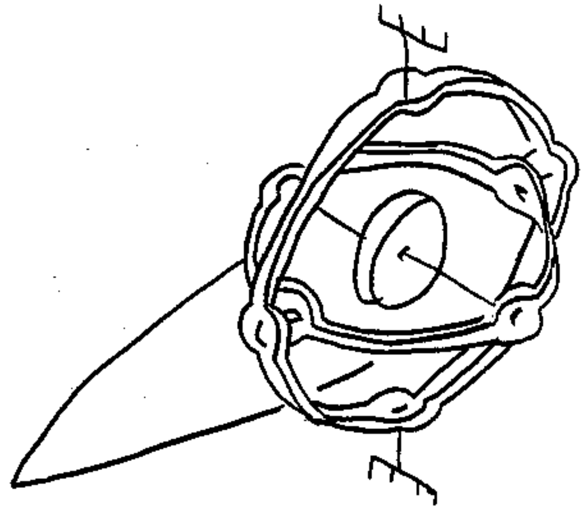
GIMBAL LOCK WHEN $G_{iV} = 90°$

$G_{iH} = 90°$

FIG. 8-11.



B



D

FIGURE 8-1-B ($\overline{B}_i = M_P M_R M_Y \overline{P}_i$) AND FIGURE 8-1-D ($\overline{B}_i = M_Y M_R M_P \overline{P}_i$)

$$\left.\begin{array}{l} G_{iV} = \phi \\ G_{oV} = \theta \end{array}\right\} \text{FOR SEQUENCE } (\psi, \phi, \theta) \qquad \left.\begin{array}{l} G_{iH} = \phi \\ G_{oH} = \psi \end{array}\right\} \text{FOR SEQUENCE } (\theta, \phi, \psi)$$

$$\bar{\Psi} = \cot^{-1}\left[\cot G_{oH}\, cG_{oV}\, cG_{iV} + (\tan G_{iH}\, sG_{iV})\csc G_{oH}\right]$$

$$\bar{\Theta} = \sin^{-1}(sG_{oV}\, cG_{iV})$$

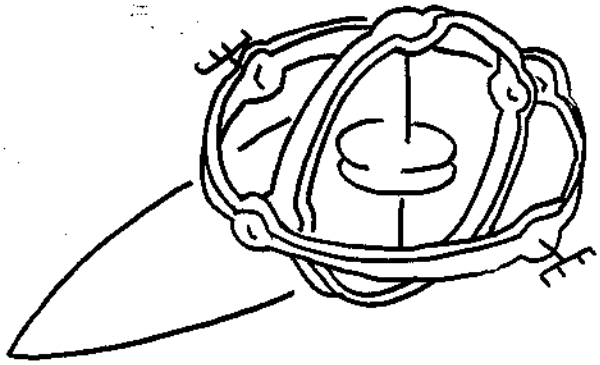
$$\bar{\Phi} = \tan^{-1}(\tan G_{iV}\, \sec G_{oV})$$

	$\overline{G}_1$	$\overline{G}_2$	$\overline{G}_3$
$\overline{B}_1$	$cG_{oH}cG_{iH}cG_{oV}cG_{iV} + sG_{iV}sG_{iH}$	$sG_{oH}cG_{iH}$	$-sG_{oV}cG_{iV}$
$\overline{B}_2$	$sG_{oH}cG_{iH}cG_{oV}cG_{iV} - sG_{oV}cG_{iV}sG_{iH}$	$cG_{oH}cG_{iH}$	sG_{iV}
$\overline{B}_3$	$sG_{oH}cG_{iH}sG_{iV} + sG_{oV}cG_{iV}cG_{oH}cG_{iH}$	$-sG_{iH}$	$cG_{oV}cG_{iV}$

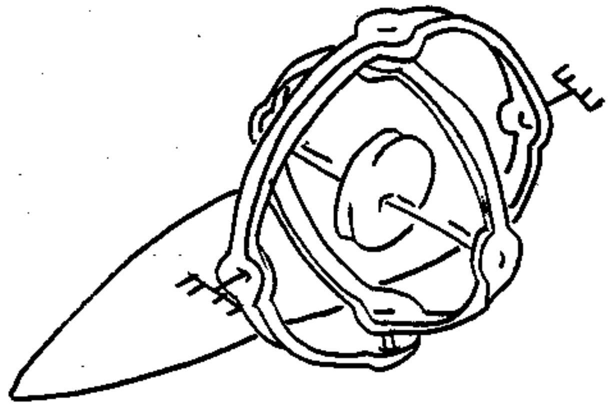
GIMBAL LOCK WHEN $G_{iV} = 90°$

$G_{iH} = 90°$

FIG. 8-12.



B



C

FIGURE 8-1-B ($\bar{B}_i = M_P M_R M_Y \bar{P}_i$) AND FIGURE 8-1-C ($\bar{B}_i = M_R M_Y M_P \bar{P}_i$)

$$\left.\begin{matrix} G_{iv} = \phi \\ G_{ov} = \theta \end{matrix}\right\}\text{FOR SEQUENCE } (\psi, \phi, \theta) \qquad \left.\begin{matrix} G_{iH} = \psi \\ G_{oH} = \phi \end{matrix}\right\}\text{FOR SEQUENCE } (\theta, \psi, \phi)$$

$$\Psi = \cot^{-1}\left[\cot G_{iH}(cG_{oH}cG_{ov}cG_{iv} + sG_{iv}sG_{oH})\right]$$

$$\Theta = \sin^{-1}(sG_{ov}cG_{iv})$$

$$\Phi = \tan^{-1}(\tan G_{iv}\sec G_{ov})$$

	$\bar{G}_1$	$\bar{G}_2$	$\bar{G}_3$
$\bar{B}_1$	$cG_{oH}cG_{iH}cG_{ov}cG_{iv} + sG_{iv}sG_{oH}cG_{iH}$	sG_{iH}	$-G_{ov}cG_{iv}$
$\bar{B}_2$	$sG_{iH}cG_{ov}cG_{iv} - sG_{ov}cG_{iv}sG_{oH}cG_{iH}$	$cG_{oH}cG_{iH}$	sG_{iv}
$\bar{B}_3$	$sG_{iH}sG_{iv} + sG_{ov}cG_{iv}cG_{oH}cG_{iH}$	$-sG_{oH}cG_{iH}$	$cG_{ov}cG_{iv}$

GIMBAL LOCK WHEN $G_{iv} = 90°$

$G_{iH} = 90°$

FIG. 8-13.

PART III GEOMETRY OF THE SPHEROID

THE GEOMETRY OF THE SPHEROID AND RELATIONSHIPS BETWEEN THE VARIOUS REFERENCE FRAMES ARE DERIVED IN THIS SECTION.

EARTH VERTICALS

THERE ARE THREE VERTICALS OF INTEREST WHEN CONSIDERING THE GEOMETRY OF AN OBLATE SPHEROIDAL EARTH, THESE ARE:

1. GEOCENTRIC VERTICAL
2. GEODETIC (PLUMB-BOB) VERTICAL
3. MASS ATTRACTION GRAVITATIONAL VERTICAL

THESE THREE VERTICALS ARE SHOWN IN FIGURE 9-1.

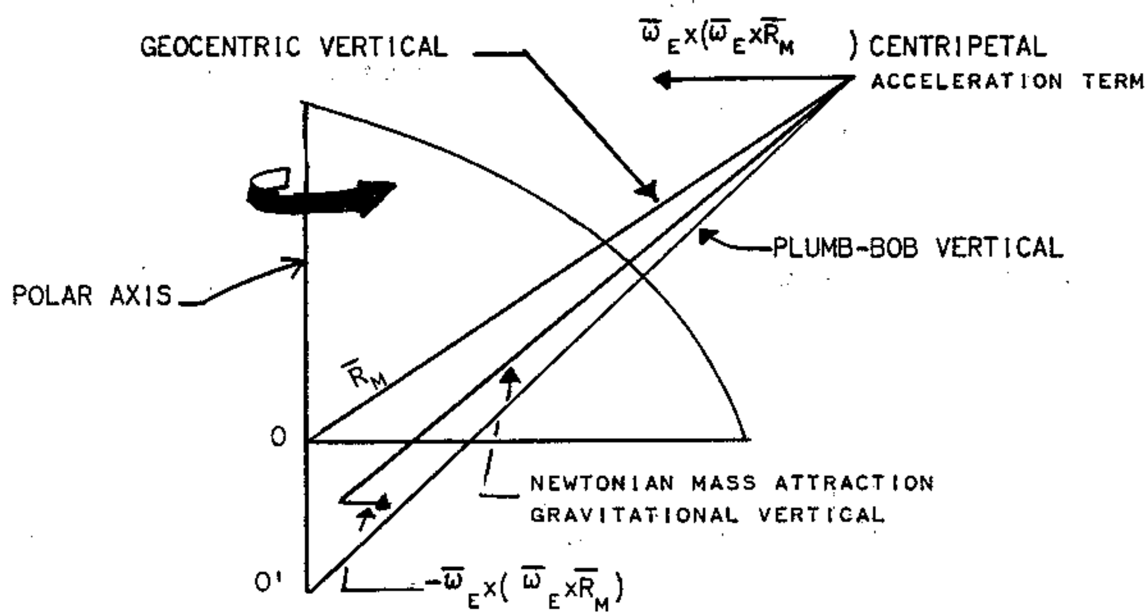


FIG. 9.1 THE THREE EARTH VERTICALS.

PLUMB-BOB VERTICALS

FOR AN OBLATE SPHEROIDAL EARTH AS SHOWN IN FIGURE 9-2 THE "EFFECTIVE GRAVITY" ACTS IN THE DIRECTION OF THE NORMAL VECTOR TO THE EARTH'S SURFACE $-\overline{G}_3$. FOR A SPHERICAL EARTH ASSUMPTION, THE PLUMB-BOB VERTICAL DOES NOT ACT ALONG THE NORMAL TO THE EARTH'S SURFACE, BUT ALONG A VECTOR PASSING TO THE SOUTH OF THE EARTH'S CENTER (IN THIS HEMISPHERE).

THE EFFECTIVE "GRAVITATIONAL FORCE" MAY BE WRITTEN AS, SEE EQUATION (11-44)

$$\overline{F}_{N_G} - M\,\overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) = \overline{F}_{GE},$$

WHERE $\overline{F}_{NG}$ IS THE NEWTONIAN MASS-ATTRACTION GRAVITY. FOR A SPHEROIDAL EARTH $\overline{F}_{N_G} - M\left\{\overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M)\right\} = -F_{GE}\,\overline{G}_3$,

AND THE LOCAL PLUMB-BOB VERTICAL WOULD BE ALONG $-\overline{G}_3$, THE NORMAL TO THE SPHEROID.

FOR A SPHERICAL EARTH THE "EFFECTIVE GRAVITY" ACTS ALONG A DIRECTION TO THE SOUTH OF THE NORMAL TO THE SURFACE AS SHOWN IN FIGURE 9-3. THUS THE PLUMB-BOB VERTICAL FOR A SPHERICAL EARTH ASSUMPTION IS NOT NORMAL TO THE TANGENT PLANE TO THE SPHERE AT THE POINT.

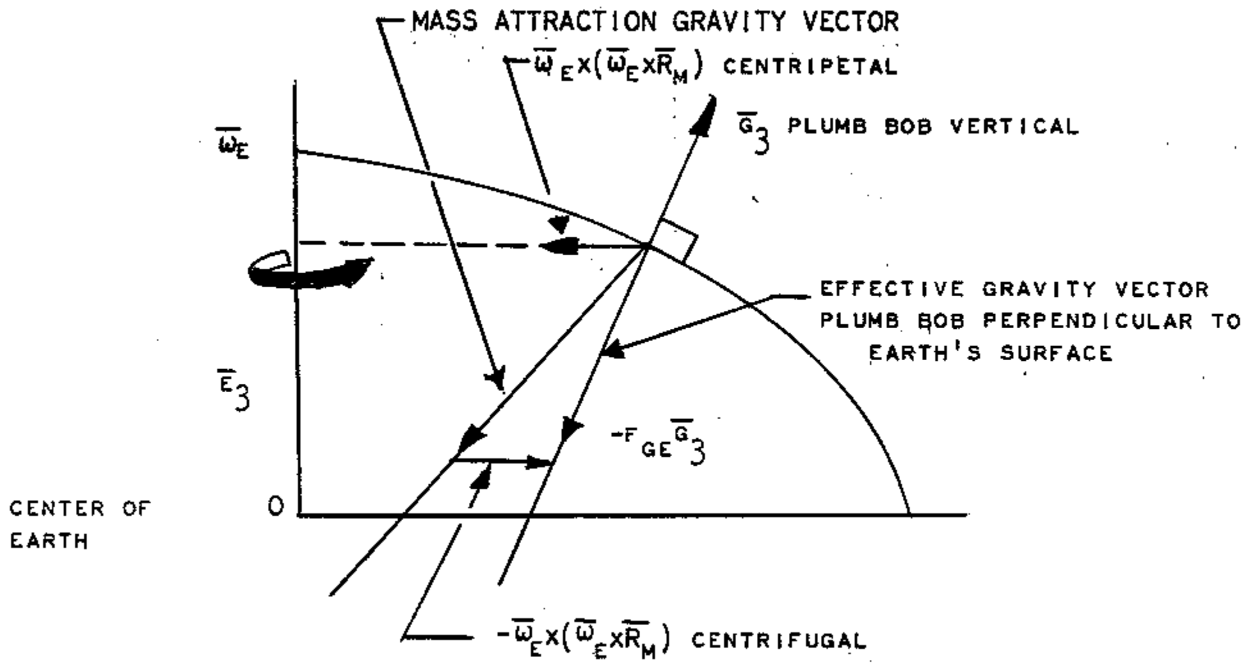


FIG. 9.2. OBLATE SPHEROIDAL (EARTH PLUMB BOB VERTICAL)

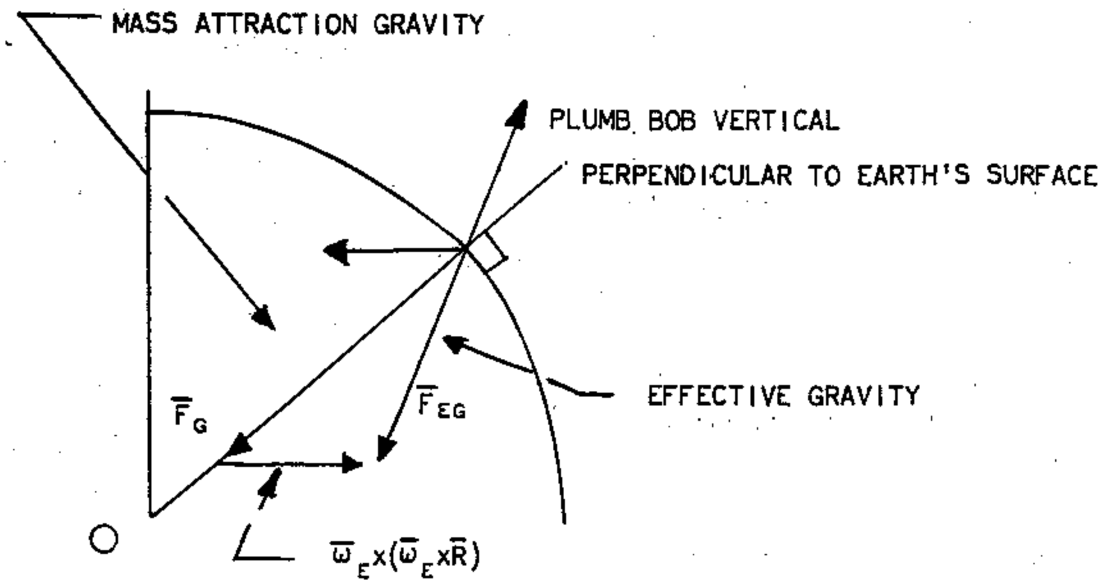


FIG. 9.3. SPHERICAL EARTH (PLUMB-BOB VERTICAL).

DISCUSSION OF EARTH COORDINATE SYSTEMS

THE EARTH IS BETTER APPROXIMATED BY AN OBLATE SPHEROID (AN ELLIPSE OF REVOLUTION ABOUT THE POLAR AXIS) THAN BY A SPHERE; CONSEQUENTLY, IT WAS FELT THAT A SPHEROIDAL COORDINATE SYSTEM RATHER THAN A SPHERICAL COORDINATE SYSTEM COULD BE USED TO ADVANTAGE. THE ANGLE BETWEEN THE NORMAL TO THE TANGENT PLANE AT A POINT ON THE SPHEROID AND THE EQUATORIAL PLANE IS GOEDETIC LATITUDE (SEE FIGURE 9-4). THESE TWO ANGLES UNIQUELY SPECIFY THE ORIENTATION OF THE NORMAL TO THE TANGENT PLANE AT A POINT ON THE SURFACE OF THE SPHEROID.

SINCE THE ALTITUDE OF ANY POINT ABOVE THE SURFACE, (OR BELOW), IS MEASURED ALONG THE NORMAL FROM THE SURFACE TO THE POINT, IT IS VERY SIMPLE TO LOCATE THE POINT IN TERMS OF ITS GEODETIC LATITUDE, LONGITUDE, HEIGHT ABOVE THE SURFACE, AND SEMI-MAJOR AND SEMI-MINOR-AXES.

IN ORDER TO LOCATE A POINT IN INERTIAL SPACE, A SET OF INERTIAL VECTORS $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ WAS CHOSEN WITH ORIGIN AT THE CENTER OF THE SPHEROID. THIS SET IS ORTHOGONAL AND RIGHT HANDED. THE $\overline{I}_3$ VECTOR WAS CHOSEN ALONG THE POLAR AXIS OF THE EARTH AND WAS DIRECTED TOWARD THE NORTH POLE. THE OTHER TWO VECTORS WERE TAKEN IN THE EQUATORIAL PLANE.

A SECOND SET $\overline{E}_I$ OF UNIT VECTORS, ALSO ORTHOGONAL AND RIGHT-HANDED, WAS FIXED TO THE ROTATING EARTH WITH ORIGIN AT THE CENTER OF THE EARTH.

THE VECTOR $\overline{E}_3$ WAS TAKEN TO BE COINCIDENT WITH $\overline{I}_3$. THE EXTENSION ALONG THE $\overline{E}_1$ VECTOR INTERSECTS THE EQUATOR AT THE GREENWICH MERIDIAN (ALTHOUGH THIS IS NOT NECESSARY). INITIALLY $\overline{I}_1 = \overline{E}_1$, $\overline{I}_2 = \overline{E}_2$; AND HENCE, AT TIME T THE ANGLE BETWEEN $\overline{I}_1$ AND $\overline{E}_1$ IS $\omega_E T$.

THE THIRD SET OF UNIT VECTORS $(\overline{H}_i)$, ALSO ORTHOGONAL AND RIGHT-HANDED, IS A SYSTEM MOVING ON THE SURFACE OF THE SPHEROID. THE TANGENTS TO LINES OF CONSTANT LATITUDE AND CONSTANT LONGITUDE AT THAT POINT GIVE THE DIRECTIONS OF LOCAL EAST AND LOCAL NORTH RESPECTIVELY. THESE TANGENTS LIE IN THE TANGENT PLANE WHICH IS PERPENDICULAR TO THE NORMAL AT THE POINT. THE SET WAS CHOSEN SO THAT: $\overline{H}_1$ IS ALONG THE TANGENT TO THE LINE OF CONSTANT LATITUDE IN THE DIRECTION OF INCREASING LONGITUDE; $\overline{H}_2$ IS ALONG THE TANGENT TO THE LINE OF CONSTANT LONGITUDE IN THE DIRECTION OF INCREASING LATITUDE; AND $\overline{H}_3$ IS DIRECTED ALONG THE OUTWARD NORMAL. THUS, $\overline{H}_1$ IS TO THE EAST AND $\overline{H}_2$ IS TO THE NORTH. THE FOURTH SET OF UNIT VECTORS USED WAS $(\overline{G}_i)$, A RIGHT-HANDED ORTHOGONAL SET. THE THREE VECTORS $\overline{G}_1$, $\overline{G}_2$, $\overline{G}_3$, ARE ORIENTED WITH RESPECT TO THE $\overline{H}_i$ FRAME THROUGH A YAW ANGLE $\psi_{G/H}$. THUS, $\overline{G}_3$ IS IN DIRECTION OF $\overline{H}_3$ (NORMAL TO TANGENT PLANE), AND THE TRANSFORMATION IS $\overline{G}_i = M_Y(\psi_{G/H})\overline{H}_i$.

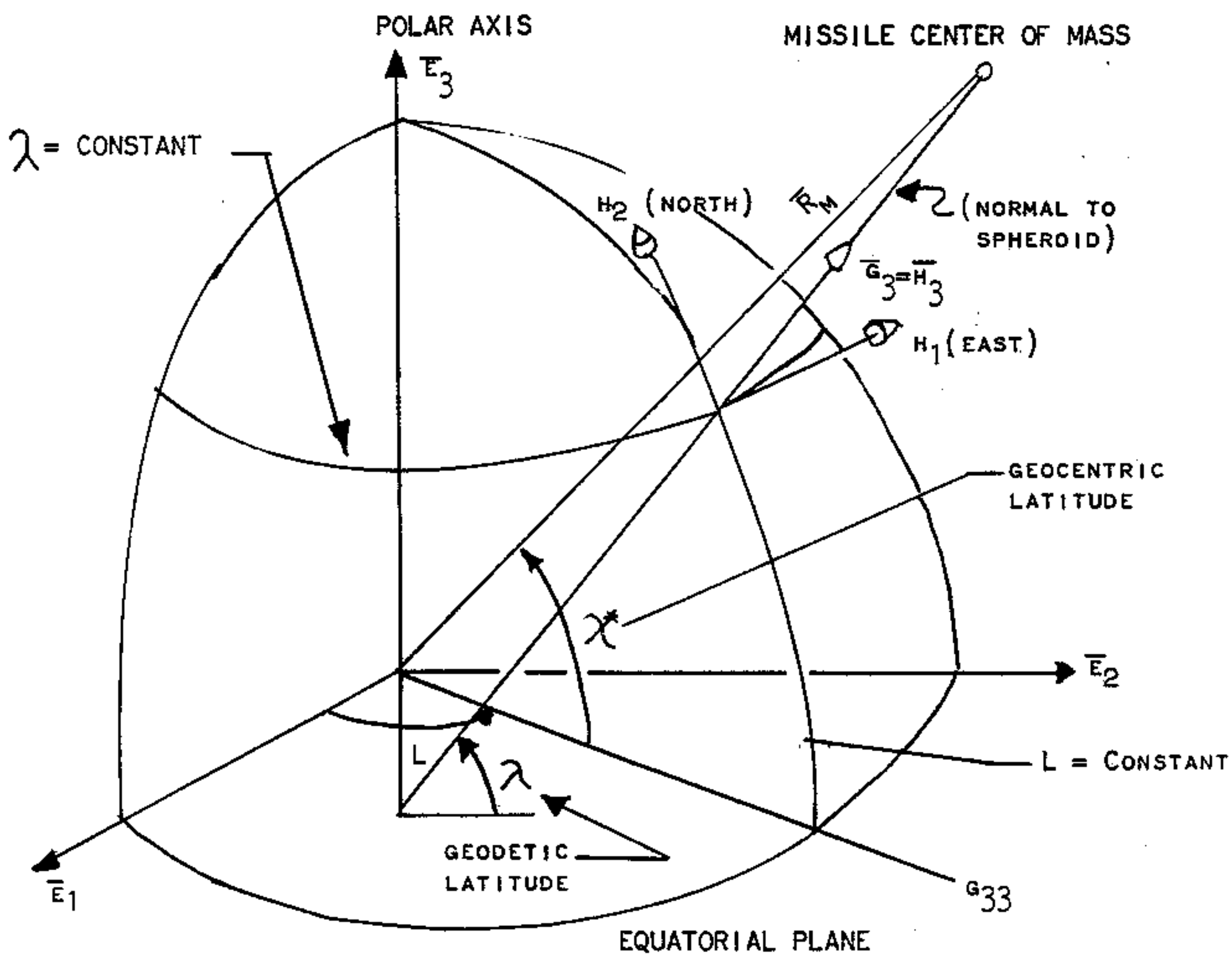


FIG. 9.4. OCTANT OF SPHEROIDAL EARTH.

ANGULAR ORIENTATION OF REFERENCE FRAMES RELATED TO THE SPHEROID

THIS TRANSFORMATION IS OBTAINED EASILY FROM FIGURE 9-5 BELOW:

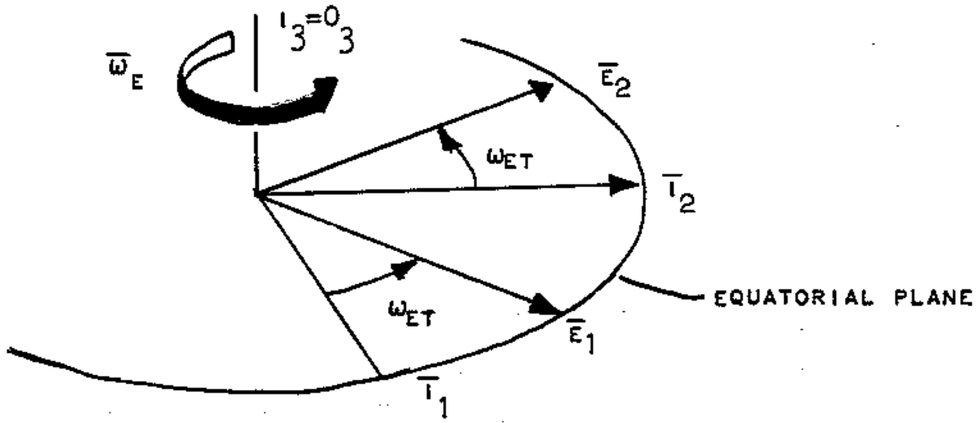


FIG. 9.5. TRANSFORMATION BETWEEN EARTH FIXED FRAME AND INERTAIL FRAME.

$$\begin{pmatrix} \overline{E}_1 \\ \overline{E}_2 \\ \overline{E}_3 \end{pmatrix} = \begin{pmatrix} \cos\omega_E T & \sin\omega_E T & 0 \\ -\sin\omega_E T & \cos\omega_E T & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{I}_1 \\ \overline{I}_2 \\ \overline{I}_3 \end{pmatrix} \qquad (9\text{-}1)$$

THE ASSUMPTION IS MADE THAT THE ANGULAR VELOCITY OF THE EARTH ABOUT ITS POLAR AXIS IS A CONSTANT.

TRANSFORMATION BETWEEN EARTH'S REFERENCE FRAME (AT CENTER OF EARTH) AND THE LOCAL HORIZONTAL FRAME FOR SPHEROIDAL EARTH AND A SPHERICAL EARTH

THE COORDINATES OF A MISSILE OR ANY OTHER POINT SUCH AS THE TARGET OR THE LAUNCH POINT ARE MOST COMMONLY KNOWN IN TERMS OF LATITUDE, LONGITUDE, AND A RADIAL DISTANCE. THE TRANSFORMATION MATRIX IS THE SAME FORM WHEN EXPRESSED IN TERMS OF GEODETIC-LATITUDE (λ) AS WHEN EXPRESSED IN TERMS OF GEOCENTRIC-LATITUDE (λ^*). THE DISTINCTION BETWEEN THE TWO LATITUDES IS DEPICTED IN FIGURE 9-4.

CONSIDER THE E_1 FRAME INITIALLY ALIGNED WITH THE G_1 FRAME. THE FIRST ROTATION IS A ROTATION OF +90° ABOUT E_2 (THE EAST VECTOR AT INTERSECTION OF EQUATOR AND GREENWICH MERIDIAN) AS SHOWN IN FIGURE 9-6.

$$\begin{pmatrix} \bar{G}_{11} \\ \bar{G}_{21} \\ \bar{G}_{31} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix} = M_P\,(90°)\,\bar{E}_1 \qquad (9\text{-}2)$$

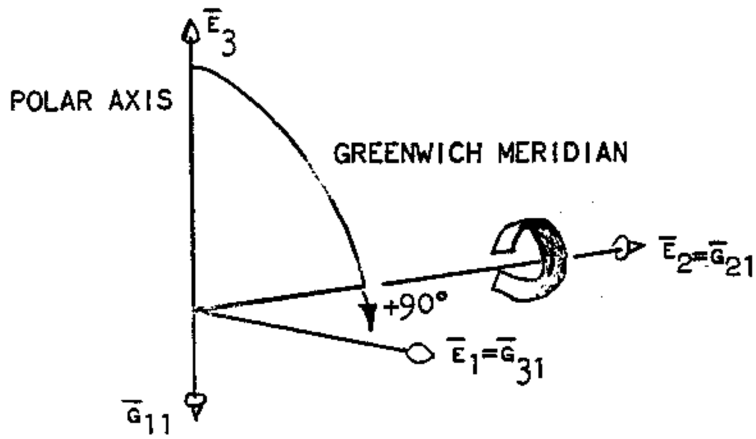


FIG. 9.6.

THE SECOND ROTATION IS A +90° ROTATION ABOUT $\bar{G}_{31}$ (THE VERTICAL AT THE INTERSECTION OF EQUATORIAL GREENWICH MERIDIAN PLANES). (SEE FIGURE 9-7.)

$$\begin{pmatrix} \bar{G}_{12} \\ \bar{G}_{22} \\ \bar{G}_{32} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{G}_{11} \\ \bar{G}_{21} \\ \bar{G}_{31} \end{pmatrix} = M_Y\,(90°)\,\bar{G}_{11} \qquad (9\text{-}3)$$

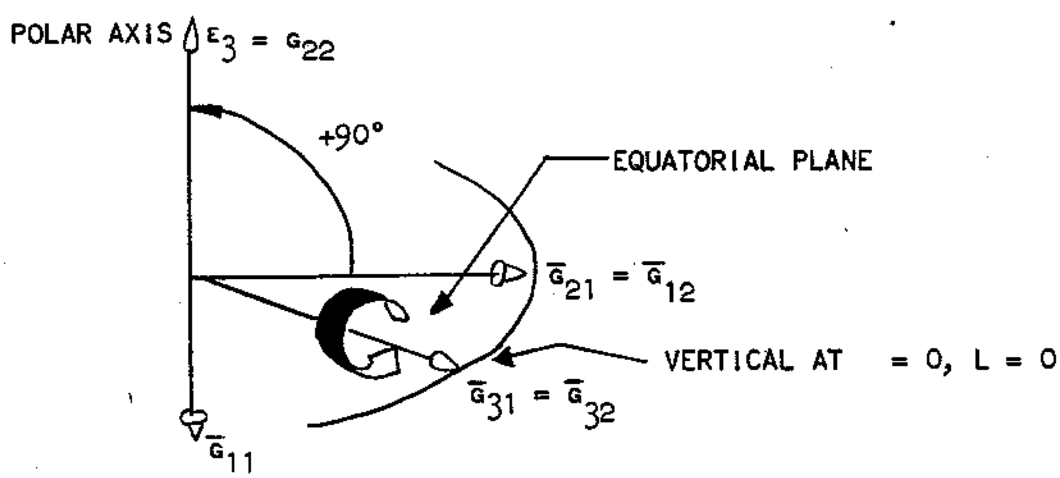


FIG. 9.7.

THE THIRD ROTATION IS A ROTATION ABOUT G_{22} (THE POLAR AXIS), FIGURE 9-6 THROUGH THE LONGITUDE ANGLE L (MEASURED POSITIVELY TOWARD THE EAST).

$$\begin{pmatrix} \bar{G}_{13} \\ \bar{G}_{23} \\ \bar{G}_{33} \end{pmatrix} = \begin{pmatrix} c\,L & 0 & -s\,L \\ 0 & 1 & 0 \\ s\,L & 0 & c\,L \end{pmatrix} \begin{pmatrix} \bar{G}_{12} \\ \bar{G}_{22} \\ \bar{G}_{32} \end{pmatrix} = M\,(L)\ \bar{G}_{12} \qquad (9\text{-}4)$$

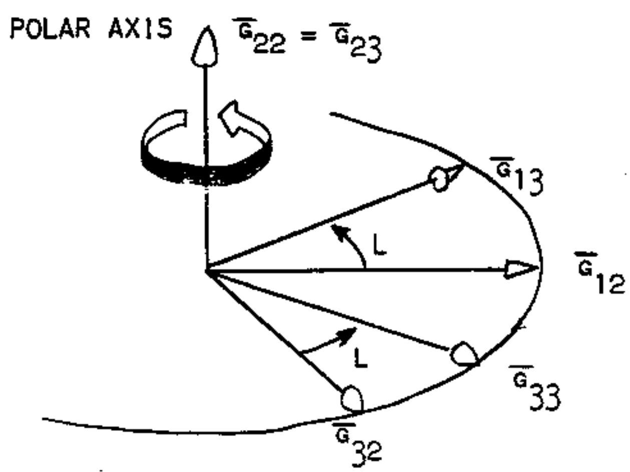


FIG. 9.8.

THE FINAL ROTATION IS A ROTATION ABOUT THE LOCAL EAST VECTOR $\bar{G}_{13}$ (SEE FIG. 9-9) THROUGH THE ANGLE.

$$\begin{pmatrix} \bar{G}_{14} \\ \bar{G}_{24} \\ \bar{G}_{34} \end{pmatrix} = \begin{pmatrix} \bar{H}_{1} \\ \bar{H}_{2} \\ \bar{H}_{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Lambda & \sin\Lambda \\ 0 & -\sin\Lambda & \cos\Lambda \end{pmatrix} \begin{pmatrix} \bar{G}_{13} \\ \bar{G}_{23} \\ \bar{G}_{33} \end{pmatrix} = M_R(\Lambda)\ \bar{G}_{13} \qquad (9\text{-}5)$$

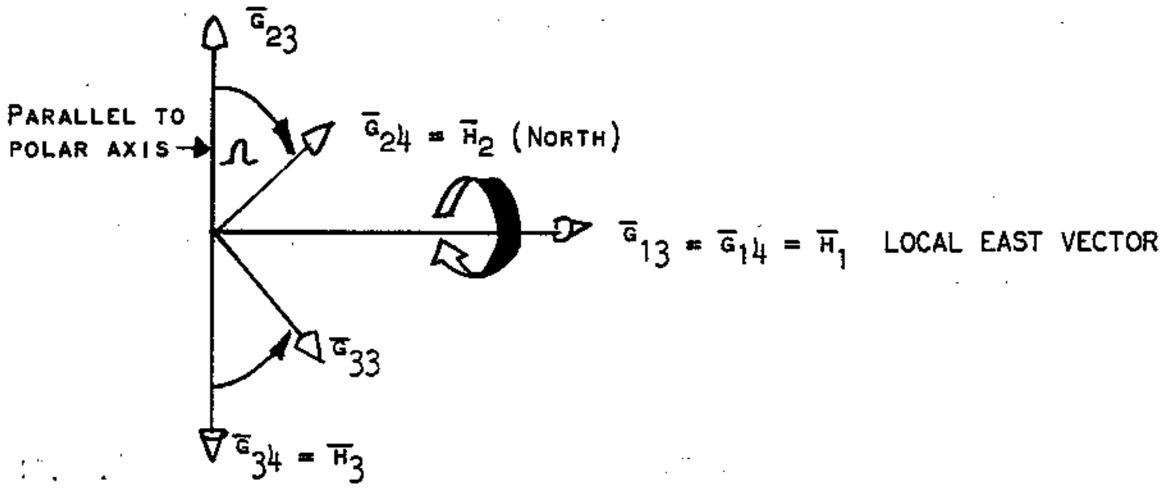


FIG. 9.9

$$\overline{H}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\Lambda & s\Lambda \\ 0 & -s\Lambda & c\Lambda \end{pmatrix} \begin{pmatrix} cL & 0 & -sL \\ 0 & 1 & 0 \\ sL & 0 & cL \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{E}_1 \\ \overline{E}_2 \\ \overline{E}_3 \end{pmatrix} \tag{9-6}$$

OR

$$\overline{H}_i = \begin{pmatrix} -sL & cL & 0 \\ cL & s\Lambda sL & c\Lambda \\ c\Lambda cL & c\Lambda sL & -s\Lambda \end{pmatrix} \begin{pmatrix} \overline{E}_1 \\ \overline{E}_2 \\ \overline{E}_3 \end{pmatrix}$$

THE FINAL ROTATION ABOUT THE LOCAL EAST VECTOR WAS TAKEN IN A POSITIVE DIRECTION, AND IF THE LATITUDE IN THE NORTHERN HEMISPHERE IS TAKEN AS POSITIVE, THEN:

$$\Lambda = -\lambda \tag{9-7}$$

SUBSTITUTING EQUATION (9-7) INTO (9-6)

$$\begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{H}_3 \end{pmatrix} = \begin{pmatrix} -s\,L & c\,L & 0 \\ -s\ \ c\,L & -s\ \ s\,L & c \\ c\ \ c\,L & c\ \ s\,L & s \end{pmatrix} \begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix} \tag{9-8}$$

$$H_I = (H^E_{IJ})\ E_I$$

IF A SPHERICAL EARTH IS CONSIDERED, THEN THE GEOCENTRIC LATITUDE IS EQUAL THE GEODETIC LATITUDE AND EQUATION (9-8), WRITTEN IN TERMS OF GEOCENTRIC LATITUDE, λ^*, IS:

$$\bar{K}_I = \begin{pmatrix} -s\,L & c\,L & 0 \\ -s\lambda^* c\,L & -s\lambda^*\ s\,L & c\lambda^* \\ c\lambda^* c\,L & c\lambda^*\ s\,L & s\lambda^* \end{pmatrix} \begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix} \tag{9-9}$$

THE FORM OF EQUATION (9-9) IS OBVIOUS FROM THE MATRIX OF EQUATION (9-8) BECAUSE THE LATTER ROTATION IS ABOUT THE LOCAL EAST VECTOR ON A SPHERICAL EARTH. IT SHOULD BE NOTED THAT THE LOCAL EAST-NORTH-VERTICAL FRAME ON A SPHERICAL EARTH IS NOT EQUAL TO THE LOCAL EAST-NORTH-VERTICAL FRAME ON A SPHEROIDAL EARTH. SINCE THE SPHEROIDAL EARTH IS FLATTENED AT THE POLES, THE ONLY MISALIGNMENT IN THE TWO FRAMES IS IN THE PLANE CONTAINING THE NORTH AND VERTICAL VECTORS. THE TWO EAST VECTORS ARE NOT AFFECTED BY THIS ELLIPTICITY.

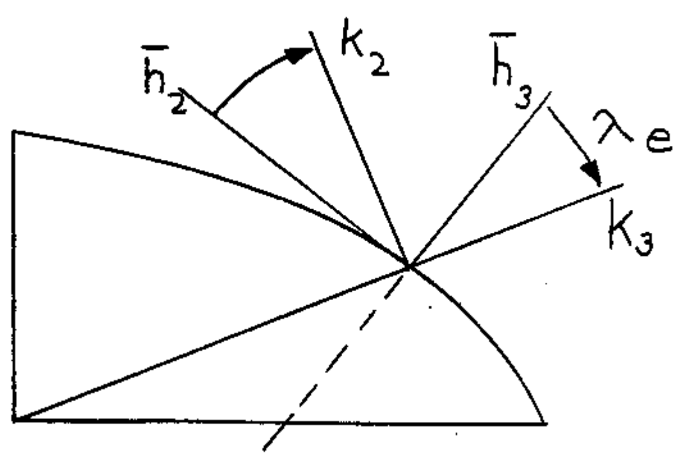


FIG. 9.10. SPHERICAL AND SPHEROIDAL EARTH VERTICAL AND NORTH FRAME.

THE TRANSFORMATION FROM THE H_I FRAME TO THE K_I FRAME IN TERMS OF THE LATITUDE ERROR $_E$ IN FIGURE 9-10) IS:

$$\bar{H}_I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\lambda_E & -s\lambda_E \\ 0 & s\lambda_E & c\lambda_E \end{pmatrix} \bar{K}_I \qquad (9\text{-}10)$$

BY EQUATIONS (9-9) AND (9-12), OMITTING THE SUBSCRIPT T, ONE OBTAINS AS THE PRODUCT OF THE TWO MATRICES,

$$\begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 \\ \bar{K}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\,(\lambda - \lambda^*) & s\,(\lambda - \lambda^*) \\ 0 & -s\,(\lambda - \lambda^*) & c\,(\lambda - \lambda^*) \end{pmatrix} \bar{H}_I$$

<u>TRANSFORMATION BETWEEN LOCAL HORIZONTAL FRAME AND OTHER KNOWN HORIZONTAL REFERENCE FRAMES SUCH AS TARGET OR LAUNCH POINT FRAME</u>

THIS TRANSFORMATION IS USEFUL WHEN CONSIDERING THE LOCATION OF THE TARGET, LAUNCH POINT, OR RADAR-SITE WITH RESPECT TO THE MISSILE. AN INERTIALLY GUIDED MISSILE MUST "KNOW" ITS COORDINATES AS WELL AS THE COORDINATES OF THE TARGET IN ORDER TO GUIDE ITSELF TO THE TARGET.

THE TRANSFORMATION BETWEEN THE EARTH'S FRAME AND THE LOCAL HORIZONTAL FRAME FOR A SPHEROIDAL EARTH IS GIVEN BY EQUATION (9-8).

$$\bar{H}_I = \begin{pmatrix} -s\,L & c\,L & 0 \\ -s\lambda c\,L & -s\lambda s\,L & c\lambda \\ c\lambda c\,L & c\lambda s\,L & s\lambda \end{pmatrix} \begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix} \qquad (9\text{-}11)$$

ATTACHING A SUBSCRIPT T TO THE ABOVE LATITUDE AND LONGITUDE COORDINATES AND INTERCHANGING ROWS AND COLUMNS, THE TRANSFORMATION BETWEEN THE TARGET HORIZONTAL REFERENCE FRAME AND THE EARTH'S FRAME IS GIVEN AS FOLLOWS:

$$\begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix} = \begin{pmatrix} -sL_T & -cL_T s\lambda_T & cL_T c\lambda_T \\ cL_T & -sL_T s\lambda_T & sL_T c\lambda_T \\ 0 & c\lambda_T & s\lambda_T \end{pmatrix} \bar{H}_{1_T} \tag{9-12}$$

THE TRANSFORMATION BETWEEN THE TARGET AND THE LOCAL HORIZONTAL FRAMES IS OBTAINED FROM EQUATIONS (9-11) AND (9-12) AS:

$$\bar{H}_1 = \begin{pmatrix} -sL & cL & 0 \\ -cLs\lambda & -sLs\lambda & c\lambda \\ cLc\lambda & sLc\lambda & s\lambda \end{pmatrix} \begin{pmatrix} -sL_T & -cL_T & s\lambda_T & cL_T c\lambda_T \\ cL_T & -sL_T & s\lambda_T & sL_T c\lambda_T \\ 0 & c\lambda_T & & s\lambda_T \end{pmatrix} \bar{H}_{1_T} \tag{9-13}$$

$$\bar{H}_1 = \begin{pmatrix} c\,(L - L_T) & s\lambda_T\, s\,(L - L_T & -c\lambda_T\, s\,(L - L_T) \\ -s\, s(L - L_T) & s\lambda s\lambda_T\, c\,(L - L_T) + c\lambda c\lambda_T & -s\lambda c\lambda_T\, c(L - L_T) + c\lambda s\lambda_T \\ c\lambda s(L - L_T) & -c\lambda s\lambda_T\, c(L - L_T) + s\lambda c\lambda_T & c\lambda c\lambda_T\, c(L - L_T) + s\lambda s\lambda_T \end{pmatrix} \bar{H}_{1_T}$$

THUS, FOR EXAMPLE, WHEN THE LATITUDE AND LONGITUDE FOR THE MISSILE AND TARGET ARE EQUAL, EQUATION (9-13) BECOMES THE IDENTITY MATRIX:

$$\bar{H}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{H}_{1_T} \tag{9-14}$$

AND THE MISSILE'S LOCAL EAST-NORTH-VERTICAL VECTORS ARE ALIGNED WITH THE TARGET EAST-NORTH-VERTICAL VECTORS.

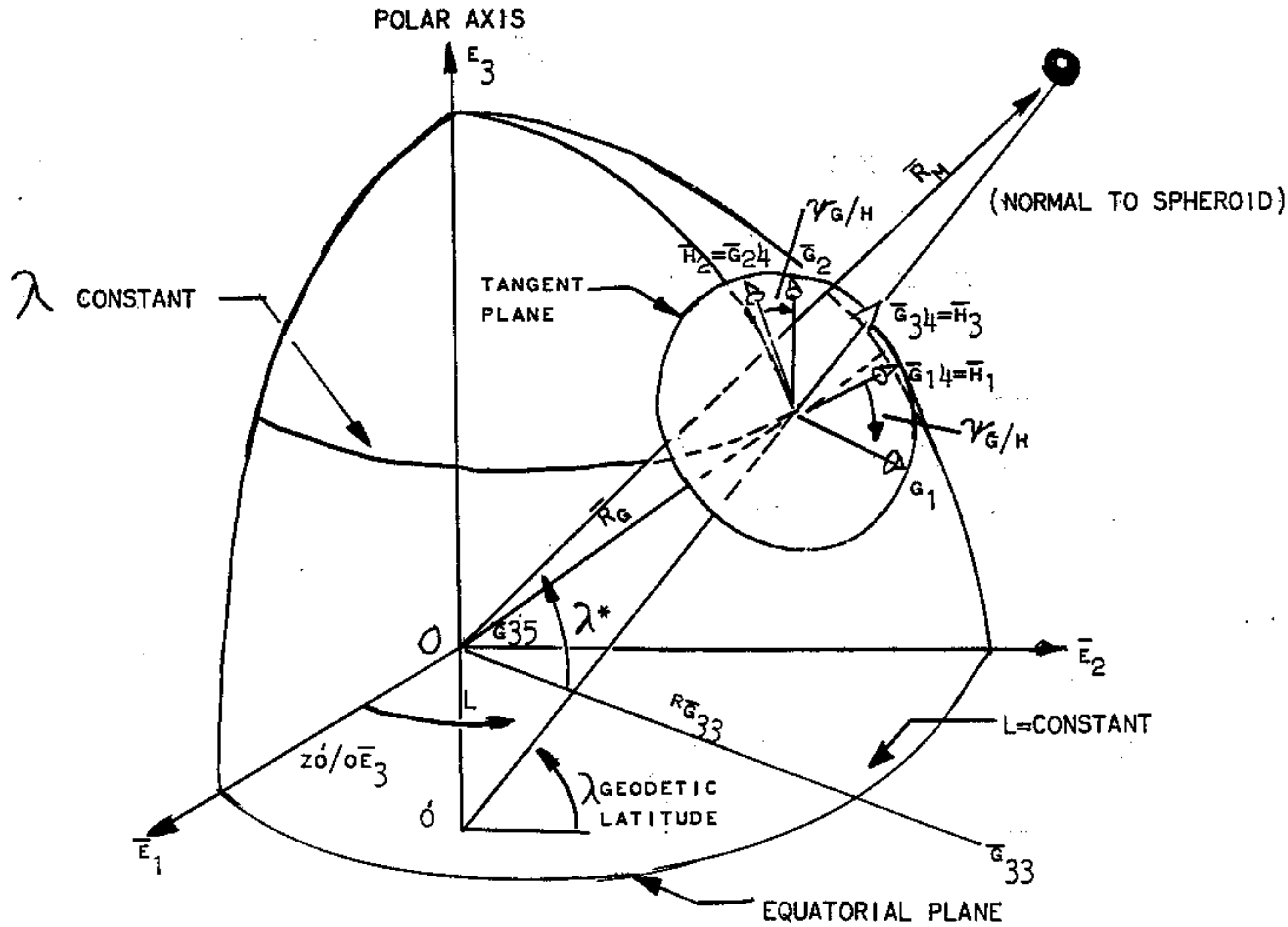


FIG. 9.10A. SPHEROIDAL EARTH GEOMETRY.

GEOMETRY

CONSIDER THE POSITION VECTOR R_G FROM THE CENTER OF THE EARTH O TO THE PROJECTION OF THE MISSILE CENTER OF MASS ONTO THE SURFACE OF THE EARTH ALONG THE LOCAL VERTICAL ($\overline{G}_3$) OF FIGURE 9.

$$\overline{R}_G = Z_{O'}^{E} \ \overline{E}_3 + R_E \ \overline{G}_3 = R_E^* \ \overline{G}_{3_S}$$

WHERE $Z_{O'}^{E}$ IS A SCALAR DISTANCE ALONG THE POLAR AXIS TO BE DETERMINED,

R_E IS THE GEODETIC EARTH-RADIUS, R^*_E IS THE GEOCENTRIC EARTH-RADIUS, AND $\overline{G}3_s$ IS THE VERTICAL TO A SPHERICAL EARTH AT THE POINT G.

INSTEAD OF SLIDING ALONG THE POLAR-AXIS TO O' AND THEN ALONG THE LOCAL PLUMB-BOB VERTICAL $\overline{G}_3$, ONE MAY SLIDE IN THE EQUATORIAL PLANE A DISTANCE R IN THE $\overline{G}_{33}$ DIRECTION AND THEN IN DIRECTION OF POLAR AXIS:

$$Z^E_G\ \overline{E}_3,\ \text{OR}\ R^*_E\ \overline{G}3_s = Z^E_{0'}\ \overline{E}_3 + R_E\ \overline{G}_3 = R\ \overline{G}_{33} + Z^E_G\ \overline{E}_3$$

$$= X^E_G\ \overline{E}_1 + Y^E_G\ \overline{E}_2 + Z^E_G\ \overline{E}_3 \tag{9-16}$$

TAKING THE SCALAR DOT PRODUCT OF EQUATION (9-16) AND EQUATION (9-15) BY $\overline{E}_3$

$$Z^E_0 = R^*_E\ \overline{G}3_s\ .\ \overline{E}_3\ \overline{G}_3\ .\ \overline{E}_3 \tag{9-17}$$

$$Z^E_0 = R^*_E\ \cos(90° - \lambda^*) - R_E\ \cos(90 - \lambda)$$

$$Z^E_0 = R^*_E\ \sin\lambda^* - R_E\ \sin\lambda \tag{9-18}$$

THE NORMAL PROJECTION OF $\overline{R}_G$ ONTO THE EQUATORIAL PLANE IS OBTAINED BY DOTTING EQUATION (9-16) BY $\overline{G}_{33}$

$$R^*_E\ \overline{G}3_s\ .\ \overline{G}_{33} = \left(Z^E_{0'}\overline{E}_3 + R_E\ \overline{G}_3\right).\ \overline{G}_{33}$$

$$R^*_E\ \overline{G}3_s\ .\ \overline{G}_{33} = R_E\ \overline{G}_3\ .\ \overline{G}_{33}\ . \tag{9-19}$$

$$R^*_E\ \cos\lambda^* = R_E\ \cos\lambda.$$

BY EQUATIONS (9-18) AND (9-19)

$$Z^E_{0'} = R_E\ \tan\lambda^*\ \cos\lambda - R_E\ \sin\lambda. \tag{9-20}$$

THE EQUATION OF THE MISSILE MERIDIAN PLANE AT THE SURFACE OF THE EARTH (AN ELLIPSE) AT POINT G IS

$$\frac{Z_G^2}{B^2} + \frac{R^2}{A^2} = 1 \tag{9-21}$$

AS SHOWN IN FIGURE 9-11.

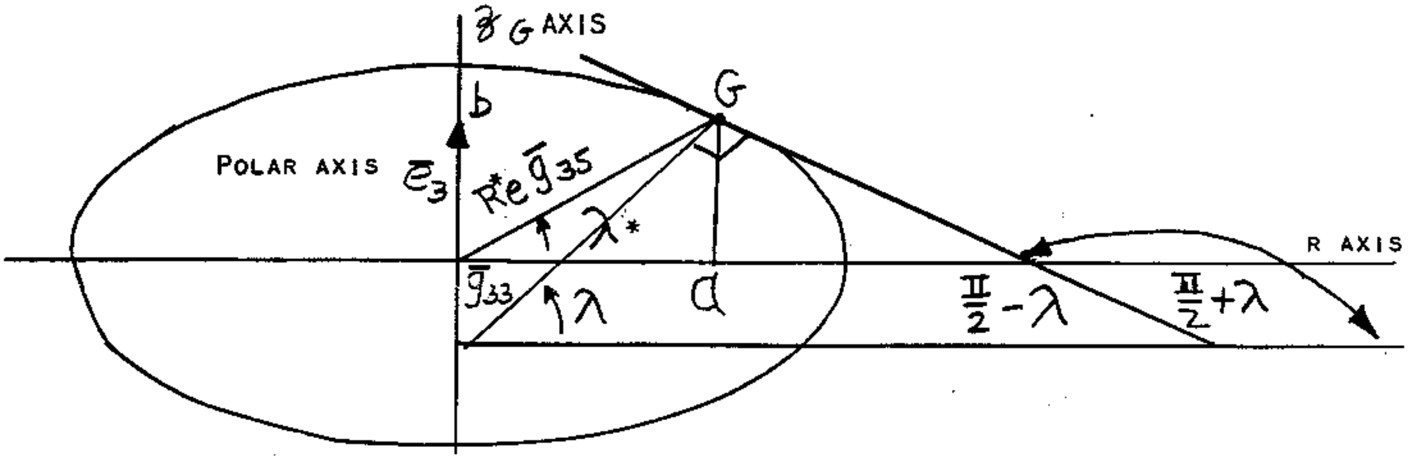


FIG. 9.11. MERIDIAN PLANE GEOMETRY.

THE SLOPE OF THE TANGENT LINE AT G IS

$$\frac{DZ_G}{DR} = \frac{-B^2}{A^2} \frac{R}{Z_G} \tag{9-22}$$

WHERE $\frac{R}{Z_G} = \cot \lambda^*$ (9-23)

AND

$$\frac{DZ_G}{DR} = \tan\left(\frac{\pi}{2} + \lambda\right) = -\cot \lambda = \frac{-B^2}{A^2} \cot \lambda^*$$

OR $\tan \lambda^* = \frac{B^2}{A^2} \tan \lambda.$ (9-24)

THUS BY EQUATION (9-20) AND EQUATION (9-24)

$$Z^E_{\sigma} = R_E \frac{B^2}{A^2} \tan\lambda \cos\lambda - R_E \sin\lambda$$

$$Z^E_{\sigma} = R_E \sin\lambda \left(\frac{B^2}{A^2} - 1\right) = - R_E \sin\lambda E^E \quad , \tag{9-25}$$

WHERE THE ECCENTRICITY E IS RELATED TO A AND B BY

$$E^2 = 1 - \frac{B^2}{A^2} \quad . \tag{9-26}$$

THE EQUATION OF THE POINT G ON THE SURFACE OF THE SPHEROID IS GIVEN BY EQUATION (9-16) AS

$$R^*_E \ \bar{G}_{3_S} = X_G \ \bar{E}_1 + Y_G \ \bar{E}_2 + Z_G \ \bar{E}_3 \quad . \tag{9-27}$$

DOTTING EQUATION (9-27) BY E_I RESPECTIVELY AND UTILIZING EQUATION (9-9), THE DIRECTION COSINE MATRIX BETWEEN THE EARTH FRAME AND A SPHERICAL EARTH GROUND FRAME, ONE OBTAINS

$$\begin{aligned} X^E_G &= R^*_E \cos\lambda^* \cos L \\ Y^E_G &= R^*_E \cos\lambda^* \sin L \\ Z^E_G &= R^*_E \sin\lambda^* \quad . \end{aligned} \tag{9-28}$$

BY EQUATIONS (9-19), (9-24) AND (9-28)

$$X^E_G = R_E \cos\lambda \cos L$$

$$Y^E_G = R_E \cos\lambda \sin L$$

$$Z^E_G = \frac{R_E \cos\lambda \sin\lambda^*}{\cos\lambda^*} = R_E \cos\lambda \tan\lambda^* =$$

$$R_E \frac{B^2}{A^2} \sin\lambda \tag{9-29}$$

UTILIZING EQUATION (9-29) WITH THE EQUATION OF THE SPHEROID IN RECTANGULAR COORDINATES

$$\frac{X_G^2 + Y_G^2}{A^2} + \frac{Z_G^2}{B^2} = 1 \tag{9-30}$$

ONE OBTAINS

$$R_E^2 \frac{(\cos^2\lambda \cos^2 L + \cos^2\lambda \sin^2 L)}{A^2} + \frac{R_E\, B^4 \sin^2\lambda}{B^2\, A^4} = 1$$

OR

$$\frac{R_E^2 \cos^2\lambda}{A^2} + \frac{R_E^2\, B^2 \sin^2\lambda}{A^4} = 1$$

FACTORING

R_E^2/A^2 ,

$$R_E^2/A^2 \left[\cos^2\lambda + \frac{B^2}{A^2} \sin^2\lambda\right] = 1$$

OR

$$R_E^2 = \frac{A^2}{\cos^2\lambda + \frac{B^2}{A^2} \sin^2\lambda}$$

BY EQUATION (9-26), $B^2/A^2 = 1 - E^2$

HENCE

$$R_E^2 = \frac{A^2}{\cos^2\lambda + \sin^2\lambda - E^2 \sin^2\lambda} = \frac{A^2}{1 - E^2 \sin^2\lambda}$$

OR

$$R_E = A\left[1 - E^2\lambda\right]^{-1/2} \tag{9-31}$$

THE ECCENTRICITY E OF EQUATION (9-26) MAY BE EXPRESSED IN TERMS OF THE ELLIPTICITY F (OR FLATTENING) OF THE EARTH BY THE RELATION GIVEN IN REFERENCE, BROWN, R.C., AND PAPPAS, J. S. "KINEMATICS OF SEVERAL STABLE PLATFORMS," CONVAIR MR-E-98, 10 SEPTEMBER 1956, FORT WORTH, TEXAS.

$$E^2 = \frac{A^2 - B^2}{A^2} = 2F - F^2 . \qquad (9\text{-}32)$$

WHERE

$$F = \frac{A-B}{A} = \frac{1}{297.00} \qquad (9\text{-}33)$$

FOR THE INTERNATIONAL SPHEROID (HAYFORD).

THE VALUES OF A (THE EQUATORAL SEMI AXIS) AND OF B (THE POLAR SEMI AXIS) ARE:

$$A = 20{,}926{,}488 \text{ FT.}$$
$$B = 20{,}856{,}029 \text{ FT.} \qquad (9\text{-}34)$$

EXPANDING EQUATION (9-31)

$$R_E = A\left[1 - E^2 \quad S^2\lambda\right]^{-1/2} = A\left[1 + \frac{E^2}{2} S^2\lambda + 3/8\ E^4\ S^4\lambda + \ldots\right], \qquad (9\text{-}35)$$

AND USING EQUATION (-32) IN (9-35) AND NEGLECTING POWERS OF F GREATER THAN ONE.

$$R_E \simeq A\left[1 + F \quad S^2\lambda\right]. \qquad (9\text{-}36)$$

AN APPROXIMATE VALUE FOR $Z^E_{O'} = - R_E \quad S \qquad E^2$ (9-37)

AND EQUATION (9-32) WITH EQUATION (9-36)

$$Z^E_{O'} = - A\left[1 + F \quad S^2\lambda\right] S\lambda\,(2F - F^2)$$

OR

$$Z^E_{O'} = -2\,A\,F \quad S\lambda, \qquad (9\text{-}38)$$

NEGLECTING POWERS OF F GREATER THAN ONE.

COMPUTING THE DIRECTION COSINES OF PLUMB-BOB VERTICAL REFERENCE FRAMES (LOCAL AND GROUND FIXED)

CONSIDER THE POSITION VECTOR OF THE MISSILE AS GIVEN BY EQUATION (10-6):

$$\overline{R}_M = Z_0^E \, \overline{E}_3 + (R_{E_M} + H_M) \, \overline{G}_3 \tag{9-39}$$

AND THE TIME DERIVATIVE OF THIS POSITION VECTOR $\overline{V}_M$,

$$\overline{V}_M = \dot{Z}_0^E \, \overline{E}_3 + (\dot{R}_{E_M} + \dot{H}_M) \, \overline{G}_3 + \omega_G \times (R_{E_M} + H_M) \, \overline{G}_3 \tag{9-40}$$

BY EQUATION (10-16)

$$\overline{V}_M = {}^{E}\overline{V}_{M/O} + \omega_E \times \overline{R}_M \, , \tag{9-41}$$

THUS EQUATING THE TWO EXPRESSIONS

$$\dot{Z}_0^E \, \overline{E}_3 + (\dot{R}_{E_M} + \dot{H}_M) \, \overline{G}_3 + \omega_G \times (R_{E_M} + H_M) \, \overline{G}_3 = {}^{E}\overline{V}_M + \omega_E \times \overline{R}_M \tag{9-42}$$

OR

$$\dot{Z}_0^E \, \overline{E}_3 + (\dot{R}_{E_M} + \dot{H}_M) \, \overline{G}_3 + \omega_G \times (R_{E_M} + H_M) \overline{G}_3 = {}^{E}\overline{V}_M + \omega_E \times Z_0^E \, \overline{E}_3 + (R_{E_M} + H_M) \, \overline{G}_3 \, . \tag{9-43}$$

THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH MAY BE EXPRESSED IN TERMS OF THE LOCAL $\overline{G}_1$ FRAME BY:

$$^{E}\overline{V}_M = \overline{V}_{M/W} + {}^{E}\overline{V}_W = {}^{R}U_M \, \overline{R}_1 + {}^{R}V_M \, \overline{R}_2 + {}^{R}W_M \, \overline{R}_3 = {}^{R}U^{G}_{i_M} \, \overline{G}_1 \tag{9-44}$$

OR DOTTING THE ABOVE EQUATION BY $\overline{G}_1$

$$\begin{aligned} {}^{R}U^{G}_{M} &= \sum {}^{R}U^{H}_{JM} \quad G^{H}_{1_J} = {}^{R}U^{G}_{M/W} + {}^{R}U^{G}_{W} \\ {}^{R}V^{G}_{M} &= \sum_{I=1}^{3} {}^{R}U^{H}_{JM} \quad G^{H}_{2_J} = {}^{R}V^{G}_{M/W} + {}^{R}V^{G}_{W} \\ {}^{R}W^{G}_{M} &= \sum {}^{R}U^{H}_{JM} \quad G^{H}_{3_J} = {}^{R}W^{G}_{M/W} + {}^{R}W^{G}_{M} \, . \end{aligned} \tag{9-45}$$

BY EQUATIONS (9-43) (9-44) AND (11-47) AND SINCE $\overline{\omega}_E \times \overline{E}_3 = \overline{0}$,

$$\dot{Z}_0^{E} \left(G_{13}^E \overline{G}_1 + G_{23}^E \overline{G}_2 + G_{33}^E \overline{G}_3 \right) + \left(\dot{R}_{E_M} + \dot{H}_M \right) \overline{G}_3 + \begin{vmatrix} \overline{G}_1 & \overline{G}_2 & \overline{G}_3 \\ P_G & Q_G & R_G \\ 0 & 0 & R_{E_M} + H_M \end{vmatrix}$$

$$= {}^{R}U_M^{G} \, \overline{G}_1 + {}^{R}V_M^{T} \, \overline{G}_2 + {}^{R}W_M^{G} \, \overline{G}_3 + \omega_E \left(R_{E_M} + H_M \right) \begin{vmatrix} \overline{G}_1 & \overline{G}_2 & \overline{G}_3 \\ G_{13}^E & G_{23}^E & G_{33}^E \\ 0 & 0 & 1 \end{vmatrix} \qquad (9\text{-}46)$$

COLLECTING COMPONENTS ALONG $\overline{G}_i$ AXES RESPECTIVELY IN THE ABOVE EQUATION:

$$\dot{Z}_0^E G_{13}^E + Q_G \left(R_{E_M} + H_M \right) \overline{G}_1 + \dot{Z}_0^E G_{23}^E - P_G \left(R_{E_M} + H_M \right) \overline{G}_2 + \dot{Z}_0^E G_{33}^E + R_{E_M} + H_M \;\; \overline{G}_3$$

$$= {}^{R}U_M^G + G_{23}^E \; \omega_E \left(R_{E_M} + H_M \right) \overline{G}_1 + \left[{}^{R}V_M^G - G_{13}^E \; \omega_E \left(R_{E_M} + H_M \right) \right] \overline{G}_2 + {}^{R}W_M^G \;\; \overline{G}_3 \qquad (9\text{-}47)$$

DOTTING THE ABOVE EQUATION BY $\overline{G}_i$ RESPECTIVELY

$$\dot{Z}_0^{E} G_{13}^E + Q_G \left(R_{E_M} + H_M \right) = {}^{R}U_M^G + G_{23}^E \;\; \omega_E \left(R_{E_M} + H_M \right) \qquad (9\text{-}48)$$

$$\dot{Z}_0^E G_{23}^E - P_G \left(R_{E_M} + H_M \right) = {}^{R}V_M^G - G_{13}^E \;\; \omega_E \left(R_{E_M} + H_M \right) \qquad (9\text{-}49)$$

$$\dot{Z}_0^E G_{33}^E + \dot{R}_{E_M} + H_M = {}^{R}W_M^G \qquad (9\text{-}50)$$

SOLVING THE TWO EQUATIONS (9-48) AND (9-49) FOR P_G AND Q_G

$$P_G = - \frac{{}^{R}V_M^G - \dot{Z}_0^{E} \; G_{23}^E}{R_{E_M} + H_M} + G_{13}^E \; \omega_E \qquad (9\text{-}51)$$

$$\bar{R}_I = (R^E_{IJ})\ \bar{E}_I = (R^E_{IJ})\ \bar{E}_J \tag{9-62}$$

$$P_R = \omega_E R^E_{13}$$

$$Q_R = \omega_E R^E_{23} \tag{9-63}$$

$$R_R = \omega_E R^E_{33}\ .$$

THE DIRECTION COSINE MATRIX OF EQUATION (9-62) MAY BE OBTAINED IN TERMS OF GEODETIC λ_R LATITUDE, LONGITUDE L_R, AND A CONSTANT YAW ANGLE $\psi_{R/E}$ MEASURED FROM THE EAST VECTOR IN THE TANGENT PLANE TO THE SPHEROID AT THE RADAR SITE, AS:

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{pmatrix} C\psi_{R/E} & S\psi_{R/E} & 0 \\ -S\psi_{R/E} & C\psi_{R/E} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -SL_R & CL_R & 0 \\ -S\lambda_R CL_R & -S\lambda_R SL_R & C\lambda_R \\ C\lambda_R CL_R & C\lambda_R SL_R & S\lambda_R \end{pmatrix} \begin{pmatrix} \bar{E}_1 \\ \bar{E}_2 \\ \bar{E}_3 \end{pmatrix}$$

$$\begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} = \begin{bmatrix} -C\psi_{R/E}\ SL_R - S\psi_{R/E}\ S\lambda_R\ CL_R & C\psi_{R/E}\ CL_R - S\psi_{R/E}\ S\ {}_R\ SL_R & S\psi_{R/E}\ C\lambda_R \\ S\psi_{R/E}\ SL_R - C\psi_{R/E}\ S\lambda_R\ CL_R & -S\psi_{R/E}\ CL_R - C\psi_{R/E}\ S\ {}_R\ SL_R & C\psi_{R/E}\ C\lambda_R \\ C\lambda_R\ CL_R & C\lambda_R\ SL_R & S\lambda_R \end{bmatrix} \bar{E}_I \tag{9-64}$$

THE DIRECTION COSINE MATRIX BETWEEN THE $\bar{G}_I$ FRAME AND THE LOCAL EAST, NORTH, VERTICAL FRAME $\bar{H}_I$ IS:

$$Q_G = \frac{R U_M^G - \dot{Z}_0^E \quad G_{13}^E + G_{23}^E \omega_E}{R_{E_M} + H_M} \qquad (9\text{-}52)$$

THE VALUE OF Z_0^E AND R_E IS GIVEN BY EQUATION (9-38) AND EQUATION (9-36) AS:

$$Z_0^E = -2 A F G_{33}^E \qquad (9\text{-}53)$$

$$R_E = A \left[1 + F \left(G_{33}^E\right)\right]^2 \qquad (9\text{-}54)$$

THE TIME DERIVATES OF EQUATION (9-53) AND EQUATION (9-54) ARE:

$$\dot{Z}_0^E = -2 A F \dot{G}_{33}^E \qquad (9\text{-}55)$$

$$\dot{R}_{E_M} = 2 A F G_{33}^E \dot{G}_{33}^E . \qquad (9\text{-}56)$$

EQUATIONS (9-51) AND (9-52) ARE THE TWO REQUIRED INERTIAL ANGULAR VELOCITIES OF THE LOCAL GROUND FRAME TO KEEP $\overline{G}_3$ ALONG THE VERTICAL; IN OTHER WORDS, THESE ARE THE TWO COMPONENTS OF INERTIAL ANGULAR VELOCITY OF THE TANGENT PLANE TO THE SPHEROID WHICH ARE ABOUT THE TWO ORTHOGONAL AXES LYING IN THE TANGENT PLANE.

THE $\overline{G}_i$ FRAME HAS BEEN DEFINED SUCH THAT $\overline{G}_3$ LIES ALONG THE PLUMB-BOB VERTICAL AND $\overline{G}_1$ AND $\overline{G}_2$ LIE IN THE TANGENT PLANE. THE INERTIAL ANGULAR VELOCITY OF $\overline{G}_i$ IS

$$\overline{\omega}_G = P_G \overline{G}_1 + Q_G \overline{G}_2 + R_G \overline{G}_3 \qquad (9\text{-}57)$$

SPECIFYING R_G WILL DEFINE WHERE THE $\overline{G}_1$ AND $\overline{G}_2$ VECTORS ARE IN THE TANGENT PLANE, AND SINCE THIS ANGULAR VELOCITY COMPONENT IS INDEPENDENT OF THE LOCAL VERTICAL, ARBITRARILY LET $R_G = 0$. (9-58)

THESE EQUATIONS (9-51), (9-50) AND (9-58) GIVE THE INERTIAL ANGULAR VELOCITIES OF THE $\overline{G}_i$ FRAME, THESE ARE:

$$P_G = \frac{-{}^R V_M^G - 2\,A\,F\,\dot{G}_{33}^E\;G_{23}^E + G_{13}^E\;\omega_E}{A\,(1 + F\,G_{33}^E) + H_M} \qquad (9\text{-}59)$$

$$Q_G = \frac{{}^R U_M^G + 2\,A\,F\,\dot{G}_{33}^E \qquad + G_{23}^E\;\omega_E}{A\,(1 + F\,G_{33}^E) + H_M}$$

$$R_G = 0.$$

THE DIRECTION COSINES G_{i3}^E (i = 1, 2, 2) OF EQUATION (9-59) MAY BE OBTAINED FROM THREE OF THE NINE FIRST ORDER DIFFERENTIAL EQUATIONS OF EQUATIONS (4-12), (4-13) AND (4-16).

THE THREE EQUATIONS ARE:

$$\dot{G}_{13}^E = -\,G_{33}^E\quad Q_G$$

$$\dot{G}_{23}^E = G_{33}^E\quad P_G$$

$$\dot{G}_{33}^E = G_{13}^E\quad Q_G - G_{23}^E\quad P_G\ . \qquad (9\text{-}60)$$

THE DIRECTION COSINE MATRIX BETWEEN THE RADAR FRAME $\overline{R}_i$ AND $\overline{G}_i$ MAY BE OBTAINED IN A SIMILAR MANNER. THE INERTIAL ANGULAR RATES OF THE RADAR FRAME ARE GIVEN AS:

$$\overline{\omega}_R = P_R\,\overline{R}_1 + Q_R\,\overline{R}_2 + R_R\,\overline{R}_3 = \overline{\omega}_E = \overline{\omega}_E\;\overline{E}_3\,, \qquad (9\text{-}61)$$

DOTTING BY $\overline{R}_i$ AND DESIGNATING THE DIRECTION COSINE MATRIX BETWEEN THE EARTH FRAME AND THE RADAR FRAME AS:

$$\bar{G}_I = M_Y \left(\psi_{G/H} \right) \bar{H}_I \,.$$

OR

$$\bar{G}_I = \begin{pmatrix} C\psi_{G/H} & S\psi_{G/H} & 0 \\ -S\psi_{G/H} & C\psi_{G/H} & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{H}_I \tag{9-65}$$

THE DIRECTION COSINE MATRIX BETWEEN THE $\bar{H}_I$ FRAME AND THE $\bar{E}_I$ FRAME IS

$$\bar{H}_I = \left(H^E_{IJ} \right) \bar{E}_J \tag{9-66}$$

OR IN TERMS OF LATITUDE λ AND LONGITUDE L,

$$\bar{H}_I = \begin{pmatrix} -SL & CL & 0 \\ -S\lambda CL & -S\lambda SL & C\lambda \\ C\lambda CL & C\lambda SL & S\lambda \end{pmatrix} \bar{E}_I \tag{9-67}$$

BY EQUATIONS (9-65) AND (9-67)

$$\bar{G}_I = \begin{pmatrix} C\psi_{G/H} & S\psi_{G/H} & 0 \\ -S\psi_{G/H} & C\psi_{G/H} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -SL & CL & 0 \\ -S\lambda CL & -S\lambda SL & C\lambda \\ C\lambda CL & C\lambda SL & S\lambda \end{pmatrix} E_I \tag{9-68}$$

MULTIPLYING THE MATRICES OF EQUATION (9-68)

$$\begin{pmatrix} -C\psi_{G/H}\, SL - S\psi_{G/H}\, S\lambda CL & CLC\psi_{G/H} - S\psi_{G/H}\, S\lambda SL & S\psi_{G/H}\, C\lambda \\ SLS\psi_{G/H} - S\lambda CL\, C\psi_{G/H} & -CLS\psi_{G/H} - C\psi_{G/H}\, S\lambda SL & C\lambda C\psi_{G/H} \\ C\lambda CL & C\lambda SL & S\lambda \end{pmatrix} \bar{E}_I \tag{9-69}$$

OR BY (9-69) $\overline{G_I} = (G^E_{IJ})\ \overline{E_J}$ (9-70)

DIVIDING G^E_{13} BY G^E_{23} OF EQUATION (9-69) ONE OBTAINS

$$\frac{S\psi_{G/H}\ C\lambda}{C\psi_{G/H}\ C\lambda} = \frac{G^E_{13}}{G^E_{23}} = \tan\psi G/H \tag{9-71}$$

OR

$$\psi_{G/H} = \text{ARC TAN}\ \frac{G^E_{13}}{G^E_{23}}. \tag{9-72}$$

IF THE LONGITUDE L IS DESIRED THEN: $L = \text{ARC TAN}\ \dfrac{G^E_{23}}{G^E_{31}}$, (9-73)

WHERE L IS NOT DEFINED WHEN = ±90°. EVEN THOUGH SOLUTIONS FOR L IN TERMS OF THE DIRECTION COSINES G^E_{IJ} DO NOT EXIST WHEN = ±90°, THIS DOES NOT ~~MEAN THAT THE~~ G^E_{IJ} DO NOT EXIST FOR THESE VALUES OF λ.

THE QUADRANTS IN WHICH $\psi_{G/H}$ AND L' WILL LIE CAN, OF COURSE, BE DETERMINED FROM THE SIGNS OF THE DIRECTION COSINES G^E_{13}, G^E_{23}, G^E_{31} AND G^E_{32} OF EQUATIONS (9-70) AND (9-73).

FOR IF λ IS NOT ±90°, THEN BY EQUATION (9-72)

$$\overline{E}_3 = G^E_{13}\ \overline{G}_1 + G^E_{23}\ \overline{G}_2 + G^E_{33}\ \overline{G}_3$$

AND DOTTING ABOVE EQUATION BY $\overline{E}_3$

$$1 = (G^E_{13})^2 + (G^E_{23})^2 + (G^E_{23})^2 \tag{9-74}$$

OR

$$(G^E_{13})^2 + (G^E_{23})^2 = 1 - \sin^2\lambda = \cos^2\lambda$$

OR

$$\left[(G^E_{13})^2 + (G^E_{23})^2 \right]^{1/2} = \cos \lambda . \tag{9-75}$$

THUS, BY EQUATIONS (9-74) AND (9-75) AND FIGURE 9-10A

$$\sin \psi_{G/H} = G^E_{13} \sec \lambda \tag{9-76}$$

AND $$\cos \psi_{G/H} = G^E_{23} \sec \lambda . \tag{9-77}$$

IN A SIMILAR MANNER

$$\sin L = G^E_{32} \sec \lambda \tag{9-78}$$

$$\cos L = G^E_{31} \sec \lambda . \tag{9-79}$$

HENCE THE ABOVE EQUATIONS ARE USED TO DETERMINE BOUNDS ON $\psi_{G/H}$ AND L.

<u>LATITUDE AND LONGITUDE ANGULAR RATES OF MISSILE</u>

A SECOND METHOD FOR THE DETERMINATION OF $\overline{\omega}_G$ FROM LATITUDE AND LONGITUDE RATES IS:

$$\overline{\omega}_G = \omega_E \, \overline{E}_3 + \dot{L} \, \overline{E}_3 - \dot{\lambda} \overline{H}_1 + \psi_{G/H} \, \overline{G}_3 , \tag{9-80}$$

WHERE BY EQUATIONS (11-47) AND (9-68)

$$\overline{E}_3 = G^E_{13} \, \overline{G}_1 + G^E_{23} \, \overline{G}_2 + G^E_{33} \, \overline{G}_3$$

$$\overline{H}_1 = c\psi_{G/H} \overline{G}_1 - s\psi_{G/H} \, \overline{G}_2 ,$$

OR

$$\overline{\omega}_G = \left[(\omega_E + \dot{L}) \, G^E_{13} - \dot{\lambda} c \psi_{G/H} \right] \overline{G}_1 + \left[(\omega_E + \dot{L}) \, G^E_{23} + \dot{\lambda} s \psi_{G/H} \right] \overline{G}_2$$

$$+ \left[(\omega_E + \dot{L}) \, G^E_{33} + \psi_{G/H} \right] \overline{G}_3 . \tag{9-81}$$

THUS, THE THREE COMPONENTS OF $\overline{\omega}_G$ ARE

$$
\begin{aligned}
P_G &= (\omega_E + \dot{L})\, G^E_{13} - \dot{\lambda}\, C\, \psi_{G/H} \\
Q_G &= -(\omega_E + \dot{L})\, G^E_{23} + \dot{\lambda}\, S\, \psi_{G/H} \\
R_G &= (\omega_E + \dot{L})\, G^E_{33} + \psi_{G/H}
\end{aligned}
\tag{9-82}
$$

THE THREE REPETITIVE EULER ANGULAR RATES $\dot{L}$, $\dot{\lambda}$ AND $\psi_{G/H}$ OF EQUATION (9-82) MUST BE KNOWN TO SOLVE EQUATION (9-82)

IF EQUATION (9-9) IS REWRITTEN AS:

$$
{}^E\overline{V}_M = \dot{Z}^E_\sigma\ \overline{E}_3 + (\dot{R}_{E_M} + \dot{H}_M)\ \overline{G}_3 + (\overline{\omega}_G - \overline{\omega}_E) \times (R_{E_M} + H_M)\ \overline{G}_3 \tag{9-83}
$$

WHERE

$\overline{\omega}_G - \overline{\omega}_E$ IS GIVEN BY EQUATION (9-80) AND $\overline{E}_3$ BY EQUATION (9-70),

THEN,

$$
{}^R U^G_M\ \overline{G}_1 + {}^R V^T_M\ \overline{G}_2 + {}^R W^T_M\ \overline{G}_3 = \dot{Z}^E_\sigma\ \overline{E}_3 + (\dot{R}_{E_M} + \dot{H}_M)\ \overline{G}_3 +
\begin{vmatrix}
\overline{G}_1 & \overline{G}_2 & \overline{G}_3 \\
\dot{L}\ G^E_{13} - \dot{\lambda}\, C\, \psi_{G/H} & \dot{L}\ G^E_{23} + \dot{\lambda}\, S\, \psi_{G/H} & \dot{L}\ G^E_{33} + \dot{\psi}_{G/H} \\
0 & 0 & R_{E_M} + H_M
\end{vmatrix}
$$

OR IN TERMS OF COMPONENTS

$$
\begin{aligned}
{}^R U^G_M &= Z^E_0\ G^E_{13} + (R_E + H_M)\ (\dot{L}\ G^E_{23} + \dot{\lambda}\, S\, \psi_{G/H}) \\
{}^R V^G_M &= Z^E_0\ G^E_{23} \qquad (R_E + H_M)\ (\dot{L}\ G^E_{13} - \dot{\lambda}\, C\, \psi_{G/H} \\
{}^R W^G_M &= (\dot{R}_E + \dot{H}_M) + \dot{Z}^E_\sigma\ G^E_{33}
\end{aligned}
\tag{9-84}
$$

If the relations for $Z^E_{0'}$, $\dot{Z}^E_{0'}$, R_E, $\dot{R}_E$ of equation (9-55) and equation (9-56) are used:

$$Z^E_{0'} = -2\ AF\ G^E_{33}\ ,\quad \dot{Z}^E_{0'} = -2\ AF\ \dot{G}^E_{33} = -2\ AF\ \dot{\lambda} \cos \lambda$$

$$R_E = A\left[1 + F\ G^2_{33}\right],\quad \dot{R}_E = 2\ AF\ G^E_{33}\ \dot{G}^E_{33} = 2\ AF\ \dot{\lambda}\ S\lambda\ C\lambda \qquad (9\text{-}85)$$

Then equation (9-60) became

$${}^R U^G_M = -2\ AF\ C\lambda\ G^E_{13}\dot{\lambda} + (A + AF\ G^{E2}_{33} + H_M)\ (\dot{L}\ G^E_{23} + \dot{\lambda}\ S\ \psi_{G/H})$$

$${}^R V^G_M = -2\ AF\dot{\lambda}\ C\ G^E_{23} - (A + AF\ G^{E2}_{33} + H_M)\ (\dot{L}\ G^E_{13} - \dot{\lambda}\ S\ \psi_{G/H})$$

$${}^R W^G_M = 2\ AF\dot{\lambda}\ S\lambda\ C\lambda + \dot{H}_M - 2\ AF\dot{\lambda}\ C\ \lambda\ G^E_{33} \qquad (9\text{-}86)$$

If $\psi_{G/H} = 0$, then the $\overline{G}_1$ frame equals the $\overline{H}_1$ frame and equation (9-87) becomes:

$${}^R U^H_M = -2\ AF\ C\lambda\ G^E_{13}\dot{\lambda} + (A + AF\ G^{E2} + H_M)\ \dot{L}\ G^E_{23}$$

$${}^R V^H_M = -2\ AF\dot{\lambda}\ C\ \lambda\ G^E_{23} - (A + AF\ G^{E2}_{33} + H_M)\ (\dot{L}\ G^E_{13} - \dot{\lambda})$$

$${}^R W^H_M = 2\ AF\dot{\lambda}\ S\lambda\ C\lambda + H_M - 2\ AF\dot{\lambda}\ C\ \lambda\ G^E_{33} \qquad (9\text{-}87)$$

The equation (9-87) becomes

$$\dot{\lambda} = \frac{{}^R V^H_M}{H_M + A + AF\ (\cos^2\lambda + 1)} \qquad (9\text{-}88)$$

$$\dot{L} = \frac{{}^R U^H_M}{(A + AF\ \sin^2\lambda + H_M)\ \cos\lambda}$$

$$\psi_{G/H} = 0\ .$$

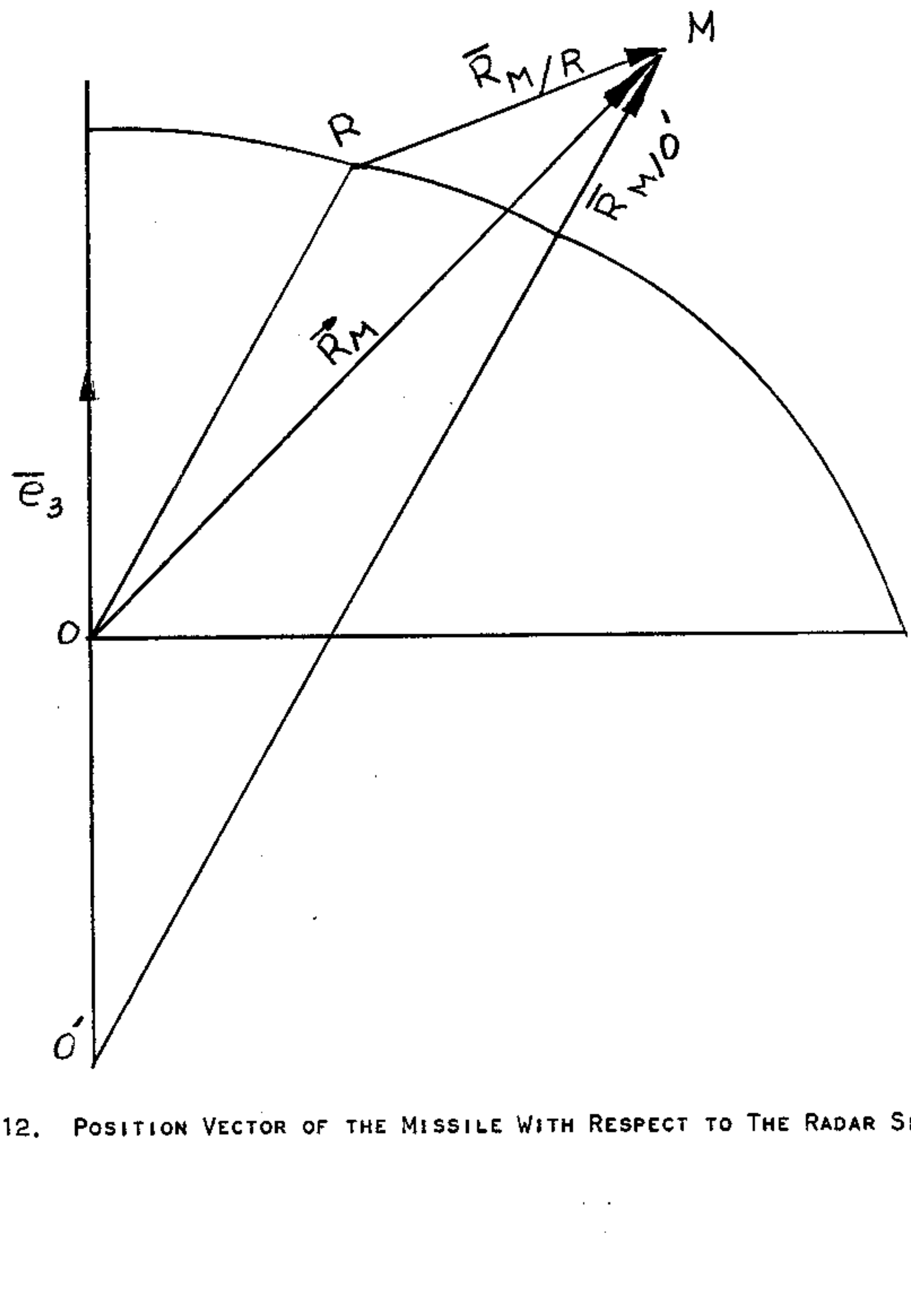


FIG. 9.12. POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE RADAR SITE.

PART IV

SELECTION OF REFERENCE FRAMES FOR SIMULATED MISSILE DYNAMICS

THE SELECTION OF REFERENCE FRAMES AND EULER SEQUENCES SHOULD BE BASED ON THE TYPE OF MISSILE UNDER STUDY. FOR EXAMPLE, THE GIMBALING REQUIREMENTS OF FREE GYROS OR 3 AXIS, 3 DEGREE OF FREEDOM STABLE PLATFORMS FOR SHORT RANGE SURFACE-TO-SURFACE OR SURFACE-TO-AIR MISSILES WOULD BE SUCH AS TO AVOID ANY EULER SEQUENCE HAVING A PITCH (A ROTATION ABOUT AN INTERMEDIATE NUMBER TWO AXIS) AS THE SECOND ANGLE OF THE SEQUENCE. AN AIR TO AIR MISSILE WOULD HAVE A DIFFERENT GIMBAL CONFIGURATION REQUIREMENT. THUS, FOR THE CLASS OF SURFACE-TO-AIR OR SURFACE-TO-SURFACE MISSILES THE EULER SEQUENCE MOST OFTEN ENCOUNTERED MAY BE PITCH, YAW, ROLL (θ, ψ, ϕ) OR PITCH, ROLL, YAW (θ, ϕ, ψ).

THE SELECTION OF THE VARIOUS REFERENCE FRAMES TO WHICH THE TRANSLATIONAL ACCELERATION, VELOCITY AND POSITION VECTORS ARE REFERRED ALSO REQUIRES A CONSIDERATION OF THE OVER-ALL SYSTEM. THE ADVANTAGE OF SEPARATING THE ANGULAR ORIENTATION OF THE VARIOUS REFERENCE FRAMES FROM THE TRANSLATIONAL WILL BE APPARENT IN THE CONSIDERATIONS TO BE DERIVED IN THIS SECTION.

A LARGE CLASS OF MISSILE SYSTEMS DEVELOPED MAY EITHER HAVE A GROUND-FIXED TRACKING DEVICE AS PART OF THE GUIDANCE SYSTEM (COMMAND GUIDANCE) OR MAY HAVE GROUND-FIXED TRACKING DATA (RADAR, OPTICAL, ETC.) AVAILABLE FOR TRAJECTORY AND PERFORMANCE DATA.

AS AN EXAMPLE OF THE VARIOUS POSSIBILITIES (POSSIBILITIES WHICH ARE USEFUL) ONE IS CONFRONTED WITH, THE REMAINING PORTION OF THIS SECTION, BRIEFLY RUNS THROUGH SOME REPRESENTATIONS OF VECTORS OF INTEREST ON THE POSITION, VELOCITY, AND FINALLY, ACCELERATION LEVEL.

MISSILE POSITION VECTORS

THE POSITION VECTOR OF THE MISSILE CENTER OF MASS (POINT M) WITH RESPECT TO THE INERTIAL POINT O AT THE CENTER OF THE EARTH, AS SHOWN IN FIGURE 10-1, MAY BE REPRESENTED IN NUMEROUS WAYS, SOME OF WHICH ARE:

$$\bar{R}_M = \bar{R}_R + \bar{R}_{M/R} \tag{10-1}$$

= POSITION VECTOR OF RADAR SITE R WITH RESPECT TO O + POSITION VECTOR OF MISSILE WITH RESPECT TO THE RADAR SITE.

$$\overline{R}_M = \overline{R}_{O'} + \overline{R}_{M/O'} \qquad (10\text{-}2)$$

= POSITION VECTOR OF GEODETIC CENTER (POINT O ') + POSITION VECTOR OF MISSILE WITH RESPECT TO GEODETIC CENTER.

$$\overline{R}_M = R_{M/O}\ \overline{K}_3 \qquad (10\text{-}3)$$

= SCALAR MAGNITUDE OF R_M ALONG THE K_3 VECTOR (VERTICAL TO A SPHERICAL EARTH).

$$\overline{R}_M = x_M^E\ \overline{E}_1 + y_M^E\ \overline{E} + z_M^E\ \overline{E}_3 \qquad (10\text{-}4)$$

= RECTANGULAR COORDINATES OF POINT M WITH RESPECT TO O ALONG THE $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$ DIRECTIONS.

$$\overline{R}_M = x_{1R/O}^E\ \overline{E}_1 + x_{1M/R}^R\ \overline{R}_1 \qquad (10\text{-}5)$$

= RECTANGULAR COORDINATES OF EARTH - FIXED RADAR SITE R WITH RESPECT TO O IN THE $\overline{E}_1$ FRAME + RECTANGULAR COORDINATES OF THE MISSILE WITH RESPECT TO THE RADAR SITE AS MEASURED IN THE RADAR FRAME. (THE LATTER REPRESENT RADAR RECTANGULAR COORDINATES).

$$\overline{R}_M = z_0^E\ \overline{E}_3 + (R_E + H_M)\ \overline{H}_3 \qquad (10\text{-}6)$$

= A SCALAR DISTANCE ALONG THE POLAR AXIS TO POINT O' + (GEODETIC EARTH RADIUS + MISSILE LOCAL ALTITUDE ALONG THE PLUMB-BOB VERTICAL.

$$\overline{R}_M = \overline{R}_R + R_{M/R}\ \overline{S}_{1_M} \qquad (10\text{-}7)$$

= SAME AS BEFORE + MISSILE RANGE WITH RESPECT TO THE RADAR SITE ALONG THE SIGHT LINE VECTOR.

MISSILE VELOCITY VECTORS

THE VELOCITY OF THE MISSILE C.G. (POINT M), WITH RESPECT TO THE INERTIAL POINT O AS OBSERVED BY AN INERTIAL OBSERVER IS THE TOTAL TIME DERIVATIVE OF THE POSITION VECTOR $\overline{R}_M$, THUS:

$$\dot{\overline{R}}_M = \overline{V}_M \qquad (10\text{-}8)$$

WHICH RESULTS FROM TAKING THE DERIVATIVE OF THE LEFT HAND SIDE OF THE ABOVE EQUATIONS. ONE MAY EXPRESS THE MISSILE VELOCITY IN MANY OTHER WAYS, SOME OF WHICH ARE:

$$\overline{V}_M = u_M^T \overline{T}_1 + 0\, \overline{T}_2 \quad 0\, \overline{T}_3 \qquad (10\text{-}9)$$

= VELOCITY OF UNIT M WITH RESPECT TO POINT O ALONG THE T_1 UNIT VECTOR DIRECTION (TANGENTIAL TO THE SPACE TRAJECTORY).

$$\overline{V}_M = u_M^I \overline{I}_1 + v_M^I \overline{I}_2 + w_M^I \overline{I}_3 \qquad (10\text{-}10)$$

= ORTHOGONAL COMPONENTS ALONG THE INERTIAL VECTORS $\overline{I}_I$ AS OBSERVED BY AN INERTIAL OBSERVER.

$$\overline{V}_M = u_M^B \overline{B}_1 + v_M^B \overline{B}_2 + w_M^B \overline{B}_3 \qquad (10\text{-}11)$$

= ORTHOGONAL COMPONENTS OF $\overline{V}_M$ ALONG INSTANTANEOUS BODY AXES AS OBSERVED BY AN INERTIAL OBSERVER.

$$\overline{V}_M = \overline{V}_{M/W} + \overline{V}_W$$

= VELOCITY OF THE MISSILE WITH RESPECT TO THE LOCAL WINDS AS OBSERVED BY AN INERTIAL OBSERVER + THE VELOCITY OF THE WINDS WITH RESPECT TO O AS OBSERVED BY AN INERTIAL OBSERVER.

THE LAST TERM $\overline{V}_W$ MAY BE FURTHER BROKEN DOWN AS

$$\overline{V}_M = \overline{V}_{M/W} + {}^R\overline{V}_{W/R} + {}^{R/}\dot{{}_I}\, \overline{V} \qquad (10\text{-}12)$$

= SAME AS ABOVE + THE VELOCITY OF THE WIND WITH RESPECT TO THE RADAR SITE (POINT R) AS OBSERVED BY AN OBSERVER FIXED TO THE EARTH AT THE RADAR SITE (I.E. TO THE RADAR FRAME $\overline{R}_I$) + THE VELOCITY OF THE RADAR SITE (DUE TO THE EARTHS ROTATION $\omega_E \times R_M = {}^{R/}\dot{{}_I}\, V$ AS OBSERVED BY AN INERTIAL OBSERVER.

ADDITIONAL EXPRESSIONS FOR THE VELOCITY OF THE MISSILE WITH RESPECT TO POINT O AS OBSERVED BY AN INERTIAL OBSERVER MAY BE OBTAINED BY TAKING THE DERIVATIVES OF THE RIGHT HAND SIDE OF EQUATIONS 10-1 THROUGH 10-6, THUS: BY EQUATION 10-1

$$\overline{V}_M = \overline{V}_R + \overline{V}_{M/R} \qquad (10\text{-}13)$$

= THE VELOCITY OF R WITH RESPECT TO POINT O AS OBSERVED BY AN INERTIAL OBSERVER + THE VELOCITY OF POINT M WITH RESPECT TO THE RADAR SITE AS OBSERVED BY AN INERTIAL OBSERVER.

BY EQUATION (10-2)

$$\overline{V}_M = \overline{V}'_O + \overline{V}_{M/O}' \tag{10-14}$$

= VELOCITY OF GEODETIC CENTER WITH RESPECT TO GEOCENTRIC CENTER + VELOCITY OF M WITH RESPECT TO GEODETIC CENTER (BOTH OBSERVED BY INERTIAL OBSERVER).

BY EQUATION (10-3)

$$\overline{V}_M = \dot{R}_M \overline{K}_3 + \overline{\omega}_{K_1} \times \overline{R}_M = {}^K\overline{V}_M + \overline{\omega}_K \times \overline{R}_M \tag{10-15}$$

= THE VELOCITY OF THE MISSILE WITH RESPECT TO POINT O AS OBSERVED BY AN OBSERVER ON THE K_1 FRAME + VELOCITY OF A POINT FIXED TO H_1 FRAME AT THE POINT M WITH RESPECT TO O OBSERVED BY AN INERTIAL OBSERVER.

BY EQUATION (10-4)

$$\overline{V}_M = {}^E\dot{x}_{1_M} \; \overline{E}_1 + \overline{\omega}_E \times \overline{R}_M = {}^E\overline{V}_M + \overline{\omega}_E \times \overline{R}_M \tag{10-16}$$

= THE VELOCITY OF POINTS M WITH RESPECT TO O AS OBSERVED BY AN OBSERVER FIXED TO THE EARTH FRAME + THE VELOCITY THE OBSERVER WOULD HAVE IF HE WERE FIXED TO THE EARTH FRAME AT A DISTANCE $\overline{R}_M$ ALONG THE $\overline{K}_3$ VECTOR AS OBSERVED BY AN INERTIAL OBSERVER.

BY EQUATION (10-5)

$$\overline{V}_M = {}^E\dot{x}_{1_R} \; \overline{E}_1 + \overline{\omega}_E \times \overline{R}_R + {}^R\dot{x}_{1_{M/R}} \; \overline{R}_1 + \overline{\omega}_R \times \overline{R}_{M/R} \tag{10-17}$$

AND ${}^E x_{1_R} = 0$, SINCE POINT R IS EARTH FIXED, THUS

$$\overline{V}_M = {}^R\overline{V}_{M/R} + \overline{\omega}_E \times (\overline{R}_R + \overline{R}_{M/R}) = {}^R\overline{V}_{M/R} + \overline{\omega}_E \times \overline{R}_M \tag{10-18}$$

= VELOCITY OF POINT M WITH RESPECT TO POINT R AS OBSERVED BY AN OBSERVER FIXED TO THE RADAR FRAME + VELOCITY OF AN EARTH FRAME FIXED POINT.

BY EQUATIONS (10-16) AND (10-18)

$$ {}^{E}\overline{V}_M = {}^{R}\overline{V}_{M/R} \tag{10-19} $$

= THAT IS, THE VELOCITY OF THE MISSILE WITH RESPECT TO THE INERTIAL ORIGIN AS OBSERVED BY AN OBSERVER FIXED TO THE EARTH FRAME IS THE SAME AS THE VELOCITY OF THE MISSILE WITH RESPECT TO THE RADAR SITE AS OBSERVED BY AN OBSERVER FIXED TO THE RADAR FRAME. THE ABOVE CONCLUSION IS TO BE EXPECTED SINCE THE RADAR AND EARTH FRAMES HAVE NO RELATIVE ANGULAR MOTION AND THE POINT R IS FIXED TO THE EARTH.

BY EQUATION (10-6)

$$ \overline{V}_M = \dot{Z}_{O'}\ \overline{E}_3 + (\dot{R}_E + \dot{H}_M)\ \overline{H}_3 + \overline{\omega}_H \times (R_E + H_M)\ \overline{H}_3 $$

$$ = \overline{V}_{O'} + {}^{H}\overline{V}_{M/O'} + \overline{\omega}_H \times (R_E + H_M)\ \overline{H}_3 \tag{10-20} $$

= VELOCITY OF O' WITH RESPECT TO O + VELOCITY OF POINT M WITH RESPECT TO O' AS OBSERVED BY AN OBSERVER FIXED TO THE $\overline{H}_I$ FRAME + VELOCITY OF A POINT (AT POINT M AND FIXED WITH RESPECT TO THE $\overline{H}_I$ FRAME) WITH RESPECT TO POINT O', AS OBSERVED BY AN INERTIAL OBSERVER.

SINCE $\overline{H}_3 = \overline{G}_3$ EQUATION (10-20) MAY BE WRITTEN AS

$$ \overline{V}_M = \overline{V}_O + {}^{H}\overline{V}_{M/O'} + \overline{\omega}_G \times (R_E + H_M)\ \overline{G}_3 , \tag{10-21} $$

ALSO,

$$ {}^{H}\overline{V}_{M/O'} = (\dot{R}_E + \dot{H}_M)\ \overline{G}_3 \tag{10-22} $$

BY EQUATION 10-7

$$ \overline{V}_M = \overline{\omega}_E \times \overline{R}_M + \dot{R}_{M/R}\ \overline{S1}_M + \overline{\omega}_S \times \overline{R}_{M/R} \tag{10-23} $$

= AS BEFORE + MISSILE RANGE RATE + THE VELOCITY OF A POINT (AT POINT M AND FIXED WITH RESPECT TO THE $\overline{S}_I$ FRAME) WITH RESPECT TO THE RADAR SITE R AS OBSERVED BY AN INERTIAL OBSERVER.

MISSILE ACCELERATION VECTORS

THE ACCELERATION OF THE MISSILE CENTER OF MASS (POINT M) WITH RESPECT TO THE INERTIAL GOECENTRIC EARTH CENTER (POINT O) AS OBSERVED BY AN INERTIAL OBSERVER IS THE TIME DERIVATIVE OF $\overline{V}_M$, OR

$$\frac{D\,\overline{V}_M}{DT} = \frac{D^2\,\overline{R}_M}{DT^2} = \overline{A}_M \,. \qquad (10\text{-}24)$$

A NUMBER OF EQUIVALENT ACCELERATION VECTORS MAY BE OBTAINED BY TAKING THE TIME DERIVATIVE OF THE VECTOR $\overline{V}_M$ AS EXPRESSED BY THE RIGHT HAND TERMS OF EQUATIONS (10-8) THROUGH (10-22).

BY EQUATION (10-10)

$$\overline{A}_M = \dot{U}_M\,\overline{T}_1 + \dot{V}_M\,\overline{T}_2 + \dot{W}_M\,\overline{T}_3 \qquad (10\text{-}25)$$

$$= A1_M\,\overline{T}_1 + A2_M\,\overline{T}_2 + A3_M\,\overline{T}_3$$

= ORTHOGONAL COMPONENTS OF THE ACCELERATION OF POINT M WITH RESPECT TO POINT O AS OBSERVED BY AN INERTIAL OBSERVER.

BY EQUATION (10-9)

$$\overline{A}_M = {}^{T}\dot{U}_M\ \overline{T}_1 + \overline{\omega}_T \times \overline{V}_M \qquad (10\text{-}26)$$

= CHANGE IN $\overline{V}_M$ VECTOR AS OBSERVED BY AN OBSERVER FIXED TO THE $\overline{T}_I$ FRAME + A COMPONENT DUE TO THE FACT THAT THE $\overline{T}_I$ FRAME IS CHANGING ITS ORIENTATION.

IT IS APPARENT THAT A VERBAL DESCRIPTION OF THE PHYSICAL SIGNIFICANCE OF EACH TERM AT THE ACCELERATION LEVEL BECOMES MORE AWKWARD AND HENCE WILL NOT BE DONE IN WHAT FOLLOWS, E.G.

$$^{BR}\dot{\overline{V}}_M = {}^{BR}\dot{U}_{I\,M/R}\quad \overline{B}_I \qquad (10\text{-}27)$$

MEANS THE TIME RATE OF CHANGE A BODY FIXED OBSERVER WOULD SEE OF THE VELOCITY OF POINT M WITH RESPECT TO POINT R AS ORIGINALLY OBSERVED BY A RADAR FIXED OBSERVER.

By equation (10-11)

$$\overline{A}_M = {}^B\dot{u}_M\ \overline{B}_1 + {}^B\dot{v}_M\ \overline{B}_2 + {}^B\dot{w}_M\ \overline{B}_3 + {}_B \times \overline{V}_M \tag{10-28}$$

$$\overline{A}_M = {}^B\dot{\overline{V}}_M + \omega_B \times \overline{V}_M$$

By equation (10-12)

$$\overline{A}_M = \dot{\overline{V}}_{M/W} + {}^{IR}\dot{\overline{V}}_{W/R} + \overline{V}_R \tag{10-29}$$

If the above vectors had been to the following basis,

$$\overline{V}^B_{M/W} = u^B_{1\,M/W}\ \overline{B}_1 \tag{10-30}$$

$$\overline{V}^R_{W/R} = \dot{u}^R_{1\,W/R}\ \overline{R}_1 \tag{10-31}$$

then one would obtain

$$\overline{A}_M = {}^B\dot{\overline{V}}_{M/W} + \overline{\omega}_B \times \overline{V}_{M/W} + {}^{RR}\dot{\overline{V}}_{W/R} + \overline{\omega}_R \times {}^R\overline{V}_{W/R} + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_R). \tag{10-32}$$

By equation (10-13)

$$\overline{A}_M = \dot{\overline{\omega}}_R \times \overline{R}_R + {}^B\dot{\overline{V}}_{M/R} + \overline{\omega}_B \times \overline{V}_{M/R} \tag{10-33}$$

By equation (10-15)

$$\overline{A}_M = {}^K\ddot{R}_M\ \overline{K}_3 + \overline{\omega}_K \times {}^K\overline{V}_M + {}_H\overline{\omega}_K \times \overline{R}_M + \overline{\omega}_K \times \boxed{{}^K\overline{V}_M + \overline{\omega}_K \times \overline{R}_M} \tag{10-34}$$

$$\overline{A}_M = {}^KR_M\ \overline{K}_3 + 2\,\overline{\omega}_K \times {}^K\overline{V}_M + {}^H\overline{\omega}_K \times \overline{R}_M + \overline{\omega}_K \times (\overline{\omega}_K \times \overline{R}_M) \tag{10-34}$$

By equation (10-18)

$$\overline{A}_M = {}^{RR}\ddot{x}_{1\,M/R}\ {}^R{}_1 + \overline{\omega}_E \times {}^R\overline{V}_{M/R} + \overline{\omega}_E \times ({}^R\overline{V}_{M/R} + \overline{\omega}_E \times \overline{R}_M)$$

$$A_M = {}^{RR}\dot{\overline{V}}_{M/R} + 2\,\overline{\omega}_E \times {}^R\overline{V}_{M/R} + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) \tag{10-35}$$

= Apparent + Coriolis + Centripetal

ACCELERATION + ACCELERATION + ACCELERATION

COMPONENT COMPONENT COMPONENT

BY EQUATION (10-20)

$$\overline{A}_M = \ddot{z}^E_o \overline{E}_3 + (\ddot{R}_E + \ddot{H}_M) \overline{H}_3 + \overline{\omega}_H \times (\dot{R}_E + \dot{H}_M) \overline{H}_3 \tag{10-36}$$

$$+ {}^H\dot{\overline{\omega}}_H \times (\dot{R}_E + \dot{H}_M) \overline{H}_3 + \overline{\omega}_H \times (R_E + H_M) \overline{H}_3 + (R_E + H_M) \overline{H}_3 = \ddot{z}^E_o \overline{E}_3 +$$

$$(\ddot{R}_E + \ddot{H}_M) \overline{H}_3 + 2\omega_H \times (\dot{R}_E + \dot{H}_M) \overline{H}_3 + {}^H\dot{\overline{\omega}}_H \times (R_E + H_M) \overline{H}_3 + \left[\overline{\omega}_H \times (R_E + H_M)\right] \overline{H}_3$$

BY EQUATION (10-21)

$$\overline{A}_M = \ddot{Z}^E_o \overline{E}_3 + (\ddot{R}_E + \ddot{H}_M) \overline{H}_3 + 2\overline{\omega}_H \times (\dot{R}_E + \dot{H}_M) \overline{H}_3 \tag{10-37}$$

$$+ \dot{\overline{\omega}}_H \times (R_E + H_M) \overline{H}_3 + \overline{\omega}_H \times \left[\overline{\omega}_H \times (R_E + H_M)\right] \overline{H}_3$$

BY EQUATION (10-23)

$$\overline{A}_M = \overline{\omega}_E \times \left[\overline{\omega}_E \times \overline{R}_M + R_{M/R} \overline{S}_{1_M} + \overline{\omega}_S \times \overline{R}_{M/R}\right]$$

$$+ \ddot{R}_{M/R} \overline{S}_{1_M} + \overline{\omega}_S \times \dot{R}_{M/R} \overline{S}_1 + {}^H\dot{\overline{\omega}}_S \times \overline{R}_{M/R}$$

$$+ \overline{\omega}_S \times \left[\dot{R}_{M/R} \overline{S}_1 + \overline{\omega}_S \times \overline{R}_{M/R}\right]$$

$$= \ddot{R}_{M/R} \overline{S}_{1_M} + 2\overline{\omega}_S \times \dot{R}_{M/R} \overline{S}_{1_M} + {}^H\dot{\overline{\omega}}_S \times \overline{R}_{M/R}$$

$$+ \overline{\omega}_S \times (\overline{\omega}_S \times \overline{R}_{M/R}) \tag{10-38}$$

THE TERM ${}^H\dot{\overline{\omega}}_S$ OF EQUATION (10-38) IS NOT INDEPENDENT OF THE REFERENCE FRAME FROM WHICH THE OBSERVATION IS MADE, HOWEVER, IT IS THE SAME AS IF THE OBSERVER WERE AN INERTIAL FRAME, FOR EXAMPLE CONSIDER THE TERM $\overline{\omega}_S \times \overline{R}_{M/R}$. IF WE EXPRESS THESE VECTORS IN THE $\overline{S}_i$ AND $\overline{B}_i$ FRAMES AND EQUATE THE VECTORS

$$\overline{\omega}_S \times \overline{R}_{M/R} = \overline{\omega}_S^S \times \overline{R}^S_{M/R} = \overline{\omega}^B_S \times \overline{R}^B_{M/R},$$

THEN TAKE THE DERIVATIVE AS OBSERVED BY INERTIAL OBSERVER .

$$\frac{D}{DT}\left(\bar{\omega}_S \times \bar{R}_{M/R}\right) = {}^{I}\dot{\bar{\omega}}_S \times \bar{R}_{M/R} + \bar{\omega}_S \times {}^{I}\dot{\bar{R}}_{M/R}$$

$$= {}^{I}\dot{\bar{\omega}}_S \times \bar{R}_{M/R} + \bar{\omega}_S \times \left[{}^{S}\dot{\bar{R}}_{M/R} + \bar{\omega}_S \times \bar{R}_{M/R}\right] \qquad (10\text{-}39)$$

THE DERIVATIVE WHEN EXPRESSED IN $\bar{S}_I$ FRAME IS

$$\frac{D}{DT}\left(\bar{\omega}_S \times \bar{R}^S_{M/R}\right) = \left({}^{S}\dot{\bar{\omega}}_S + \bar{\omega}_S \times \bar{\omega}_S\right) \times \bar{R}^S_{M/R} + \bar{\omega}_S \times \left({}^{S}\dot{\bar{R}}_{M/R} + \bar{\omega}_S \times \bar{R}_{M/R}\right) \qquad (10\text{-}40)$$

$$= {}^{S}\dot{\bar{\omega}}_S \times \bar{R}^S_{M/R} + \bar{\omega}_S \times {}^{S}\dot{\bar{R}}_{M/R} + \bar{\omega}_S \times \left(\bar{\omega}_S \times \bar{R}_{M/R}\right)$$

AND THE DERIVATIVE WHEN THE VECTORS ARE IN THE $\bar{B}_I$ FRAME IS

$$\frac{D}{DT}\left(\bar{\omega}^B_S \times \bar{R}_{M/R}\right) = \left({}^{B}\dot{\bar{\omega}}_S + \bar{\omega}_B \times \omega^B_S\right) \times \bar{R}_{M/R} + \bar{\omega}_S \times \left({}^{B}\dot{\bar{R}}_{M/R} + \bar{\omega}_B \times \bar{R}_{M/R}\right)$$

$$= {}^{B}\dot{\bar{\omega}}_S \times \bar{R}_{M/R} + \left(\bar{\omega}_B \times \bar{\omega}_S\right) \times \bar{R}_{M/R} + \bar{\omega}_S \times {}^{B}\dot{\bar{R}}_{M/R} \qquad (10\text{-}41)$$

$$+ \bar{\omega}_S \times \left(\bar{\omega}_B \times \bar{R}_{M/R}\right).$$

BY EQUATION (10-39) AND (10-40)

$${}^{S}\dot{\bar{\omega}}_S = {}^{I}\dot{\bar{\omega}}_S$$

WHICH IS TO BE EXPECTED BECAUSE ${}^{I}\dot{\bar{\omega}}_S$ MEANS THE TIME RATE OF CHANGE OF THE INERTIAL ANGULAR VELOCITY VECTOR OF THE $\bar{S}_I$ FRAME AS OBSERVED BY AN INERTIAL OBSERVER, WHEREAS THE TERM ${}^{S}\dot{\bar{\omega}}_S$ MEANS THE TIME RATE OF CHANGE OF THE INERTIAL ANGULAR VELOCITY OF THE $\bar{S}_I$ FRAME AS OBSERVED BY AN OBSERVER ON THE $\bar{S}_I$ FRAME.

HOWEVER, BY EQUATION (10-39) AND (10-40)

$${}^{B}\dot{\bar{\omega}}_S \neq {}^{S}\dot{\bar{\omega}}_S, \qquad (10\text{-}42)$$

SINCE THE TERM ON THE LEFT IS THE TIME RATE OF CHANGE OF THE INERTIAL ANGULAR VELOCITY OF THE $\overline{S}_1$ FRAME AS OBSERVED BY AN OBSERVER ON THE $\overline{B}_1$ FRAME.

COORDINATE TRANSFORMATIONS OF SOME COMPONENTS AT THE VELOCITY LEVEL WILL YIELD MORE HYBRID RESULTS. FOR EXAMPLE, CONSIDER THE INERTIAL ACCELERATION OF THE MISSILE AS GIVEN BY EQUATION (10-35). IF THE VELOCITY OF THE MISSILE WITH RESPECT TO THE RADAR SITE AS OBSERVED BY AN EARTH FIXED OBSERVER IS EXPRESSED IN COMPONENTS ALONG RADAR AXES AND ALONG BODY AXES,

THEN

$$ {}^{R}\overline{V}^{R}_{M/R} = {}^{R}\overline{V}^{B}_{M/R} \quad . \tag{10-43}$$

TAKING THE DERIVATIVE OF THE LEFT HAND SIDE OF EQUATION (10-43)

$$ \frac{D}{DT}\left({}^{R}V^{R}_{M/R}\right) = {}^{RR}\dot{\overline{V}}_{M/R} + \overline{\omega}_R \times {}^{R}\overline{V}_{M/R} \quad , \tag{10-44}$$

TAKING THE DERIVATIVE OF THE RIGHT HAND SIDE

$$ \frac{D}{DT}\left({}^{R}\overline{V}^{B}_{M/R}\right) = {}^{BR}\dot{\overline{V}}_{M/R} + \overline{\omega}_B \times {}^{R}\overline{V}_{M/R} \quad . \tag{10-45}$$

EQUATING EQUATIONS (10-44) AND (10-45)

$$ {}^{RR}\dot{\overline{V}}_{M/R} + {}^{BR}\dot{\overline{V}}_{M/R} = (\overline{\omega}_B - \overline{\omega}_R) \times {}^{R}\overline{V}_{M/R} \;, \tag{10-46}$$

AND UTILIZING EQUATION (10-46) IN EQUATION (10-35)

$$ \overline{A}_M = {}^{BR}\dot{\overline{V}}_{M/R} + (\overline{\omega}_B - \overline{\omega}_R) \times {}^{R}\overline{V}_{M/R} + 2\overline{\omega}_E \times {}^{R}\overline{V}_{M/R} $$

$$ + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) \quad . \tag{10-47}$$

If one takes the second derivative of equation (10-4)

$$\overline{A}_M = \frac{D^2 \overline{R}_M}{DT^2} = {}^{EE}\ddot{X}_{M/O}\ \overline{E}_I + 2\omega_E \times {}^{E}\overline{V}_M + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{V}_M) \qquad (10\text{-}48)$$

Apparent	+	Coriolis	+	Centripetal
Acceleration		Acceleration		Acceleration ,

the familiar expression found in classical physics texts is obtained. It is to be noted that the coriolis term of equation (10-48) is the same as the coriolis expression of equation (10-47) and of equation (10-36), since by equation (10-19)

$${}^{E}\overline{V}_M = {}^{R}\overline{V}_{M/R} \ .$$

Whether the term Coriolis is reserved for the expression involving the rotating earth frame $2\overline{\omega}_E \times {}^{E}\overline{V}_M$ is not important, the fact is that mathematically the basis vectors are rotating.

SOLUTION OF MISSILE TRANSLATIONAL DYNAMICS IN BODY AXES $\frac{D}{DT}\ \overline{V}_M^B$

The translational and rotational acceleration equations of missile motion are solved for in body axes in this section.

Translational Acceleration

Consider the inertial acceleration of the missile center of mass $\overline{A}_M$ as derived from the expression for the velocity of the missile with respect to the inertial point O as given by equation (10-11)

$$\overline{V}_M = U_M^B\ \overline{B}_1 + V_M^B\ \overline{B}_2 + W_M^B\ \overline{B}_3 \qquad (11\text{-}1)$$

THE ACCELERATION IS GIVEN BY

$$\overline{A}_M = {}^B\dot{\overline{V}}_M + \overline{\omega}_B \times \overline{V}_M \; . \qquad (11\text{-}2)$$

WRITING

$$M \quad \overline{A}_M = \sum \overline{F}$$

IN TERMS OF EQUATION (11-2)

$$M \left[{}^B\dot{\overline{V}}_M + \overline{\omega}_B \times \overline{V}_M \right] = \overline{F}^B_A + \overline{F}^B_{N_G} + \overline{F}^B_T \qquad (11\text{-}3)$$

ONE OBTAINS THE TRANSLATION EQUATIONS TO BE SOLVED IN BODY AXES AS

$$M \left({}^B\dot{U}_M + Q_B \; W^B_M - R_B \; V^B_M \right) = F^B_{1_A} + F^B_{1_{N_G}} + F^B_{1_T} \qquad (11\text{-}4)$$

$$M \left({}^B\dot{V}_M + R_B \; V^B_M - P_B \; W_M \right) = F^B_{2_A} + F^B_{2_{N_G}} + F_{2_T}$$

$$M \left({}^B\dot{W}_M + P_P \; V_M - Q_B \; U_M \right) = F^B_{3_A} + F^B_{3_{N_G}} + F^B_{3_T} \; .$$

MISSILE KINEMATICS

THE VELOCITY OF THE MISSILE WRITTEN IN TERMS OF EQUATION (10-18) IS,

$$\overline{V}_M = {}^R\overline{V}_{M/R} + \overline{\omega}_E \times \overline{R}_M \qquad (11\text{-}5)$$

THE POSITION VECTOR OF THE MISSILE $\overline{R}_M$ MAY BE WRITTEN IN TERMS OF EQUATION (10-1) AS THE POSITION VECTOR OF THE RADAR SITE PLUS THE POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE RADAR SITE.

$$\overline{R}_M = \overline{R}_R + \overline{R}_{M/R} . \qquad (11\text{-}6)$$

By Figure 11-1

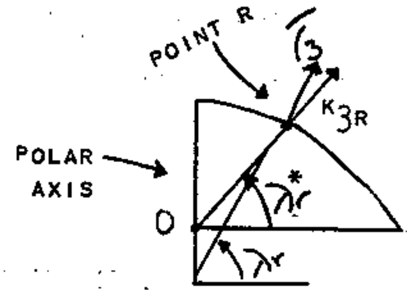


Fig. 11.1. Radar Site.

$$\overline{R}_R = R_E^* \ \overline{K}_3 = X_R^R \ \overline{R}_1 + Y_R^R \ \overline{R}_2 + Z_R^R \ \overline{R}_3 \qquad (11\text{-}7)$$

If the radar reference axes $\overline{R}_i$ are oriented in the directions of East, North and vertical axes at the point R, then

$$\overline{H}_{1_R} = \overline{R}_1 \qquad (11\text{-}8)$$

and by equation (9-10)

$$\overline{K}_{R_3} = (- c\lambda_R{}^* \ s\lambda_R + s\lambda_R{}^* \ c_R) \ \overline{R}_2 \ +$$

$$(c\lambda_R \ c\lambda_R{}^* + s\lambda_R{}^* \ s\lambda_R) \ \overline{R}_3$$

$$\overline{K}_{R_3} = K_{R_{3_2}} \ \overline{R}_2 + K_{R_{3_3}} \ \overline{R}_3 \ . \qquad (11\text{-}9)$$

where $\overline{K}_{R_{3_2}} = \overline{K}_{R_3} \ \overline{R}_2$

The relationship between geocentric latitude λ^* and geodetic latitude λ is given by equation (9-24) as,

$$\tan \lambda^* = \frac{B^2}{A^2} \ \tan \lambda . \qquad (11\text{-}10)$$

THE RELATION BETWEEN GEOCENTRIC EARTH RADIUS $R^*_{E_R}$ AT THE RADAR SITE AND GEODETIC EARTH RADIUS R_{E_R} IS GIVEN BY EQUATION (9-29) AS

$$R^*_E \sin\lambda^* = \frac{B^2}{A^2} R_E \sin\lambda \tag{11-11}$$

WHERE

$$R_E = \left(1 + F \sin^2\lambda\right) \tag{11-12}$$

THUS BY EQUATION (11-11) AND (11-12)

$$R^*_{E_R} = \frac{B^2}{A} \frac{\sin\lambda_R}{\sin\lambda^*_R} \quad (1 + F \sin^2\lambda_R) . \tag{11-13}$$

BY EQUATION (11-13) AND EQUATION (11-7), ONE OBTAINS FROM DOTTING EQUATION (11-7) BY $\overline{R}_1$ RESPECTIVELY

$$X^R_R = R_{E_R} \quad K^R_{R_{3_1}} = 0 \tag{11-14}$$

$$Y^R_R = R^*_{E_R} \quad K_{3_2} = \frac{B^2}{A} \frac{\sin\lambda_R}{\sin\lambda^*_R} \quad (1 + F \sin^2\lambda_R) \sin(\lambda_R - \lambda^*_R)$$

$$Z^R_R = R^*_{E_R} \quad K^R_{R_{3_3}} = \frac{B^2}{A} \frac{\sin\lambda_R}{\sin\lambda^*_R} \quad (1 + F \sin^2\lambda_R) \cos(\lambda_R - \lambda^*_R)$$

THE EXPRESSIONS OF EQUATION (11-14) ARE CONSTANTS FOR A GIVEN RADAR SITE. THE SECOND VECTOR OF EQUATION (11-15) MAY NOW BE WRITTEN WITH EQUATION (11-6) AND (11-14) AS:

$$\left[\overline{\omega}_E \times (\overline{R}_R + \overline{R}_{M/R})\right]^R = \omega_E \begin{vmatrix} \overline{R}_1 & \overline{R}_2 & \overline{R}_3 \\ R^E_{1_3} & R^E_{2_3} & R^E_{3_3} \\ X^R_{M/R} & Y^R_R + Y^R_{M/R} & Z^R_R + Z_{M/R} \end{vmatrix} \tag{11-15}$$

WHERE

$$\overline{\omega}_E = \omega_E E_3 = \omega_E \left({}^R1^E_3 \; \overline{R}_1 + {}^R2_3 \; R_2 + {}^R3^E_3 \; \overline{R}_3 \right). \tag{11-16}$$

IN COMPONENT FORM EQUATION (11-15) IS

$$(\overline{\omega}_E \times \overline{R}_M)^R = \omega_E \left\{ {}^R2^E_3 \left(Z^R_R + Z^R_{M/R} \right) - {}^R3^E_3 \left(Y^R_R + Y^R_{M/R} \right) \right\} \overline{R}_1$$

$$+ \omega_E \left\{ {}^R3^E_3 \; X^R_{M/R} - {}^R1^E_3 \left(Z^R_R + Z^R_{M/R} \right) \right\} \overline{R}_2 \tag{11-17}$$

$$+ \omega_E \left\{ {}^R1^E_3 \left(Y^R_R + Y^R_{M/R} \right) - {}^R2^E_3 \; X^R_{M/R} \right\} \overline{R}_3 .$$

$$= {}^{R/I}U^R_M \overline{R}_1 + {}^{R/I}V^R_M \overline{R}_2 + {}^{R/I}W^R_M \overline{R}_3 \tag{11-18}$$

AN ORTHOGONAL TRANSFORMATION FROM THE R BASIS VECTORS TO THE B BASIS VECTORS YIELDS

$${}^{R/I}U^B_M = {}^{R/I}U^R_M B^R_{11} + {}^{R/I}V^R_M B^R_{12} + {}^{R/I}W^R_M B^R_{13} \tag{11-19}$$

$${}^{R/I}V^B_M = {}^{R/I}U^R_M B^R_{21} + {}^{R/I}V^R_M B^R_{22} + {}^{R/I}W^R_M B^R_{23}$$

$${}^{R/I}W^B_M = {}^{R/I}U^R_M B^R_{31} + {}^{R/I}V^R_M B^R_{32} + {}^{R/I}W^R_M B^R_{33} .$$

EQUATING THE TWO EXPRESSIONS OF EQUATIONS (11-1) AND (11-5) FOR THE VELOCITY OF THE MISSILE WITH RESPECT TO THE INERTIAL POINT O AS OBSERVED BY AN INERTIAL OBSERVER,

$$U^B_{I_M} B_I = {}^RU^B_{I_{M/R}} B_I + (\omega_E \times R_M)^B , \tag{11-20}$$

FROM WHICH ONE OBTAINS,

$${}^RU^B_{M/R} = U^B_M - {}^{R/I}U^B_M$$

$$ {}^{R}V^{B}_{M/R} = V^{B}_{M} - {}^{R/I}\ V^{B}_{M} \quad (11\text{-}21) $$

$$ {}^{R}W^{B}_{M/R} = W^{B}_{M} - {}^{R/I}\ W^{B}_{M} \ . $$

AN ORTHOGONAL TRANSFORMATION OF THE MISSILE VELOCITY VECTOR IN THE B_I FRAME TO THE H_I FRAME YIELDS.

$$ {}^{R}U^{B}_{M/R} = {}^{R}U^{B}_{M/R}\ H^{B}_{11} + {}^{R}V^{B}_{M/R}\ H^{B}_{12} + {}^{R}W^{H}_{M/R}\ H^{B}_{13} $$

$$ {}^{R}V^{H}_{M/R} = {}^{R}U_{M/R}\ H^{B}_{21} + {}^{R}V^{B}_{M/R}\ H^{B}_{22} + {}^{R}W^{B}_{M/R}\ H^{B}_{23} \quad (11\text{-}22) $$

$$ {}^{R}W^{H}_{M/R} = {}^{R}U^{B}_{M/R}\ H^{B}_{31} + V^{B}_{M/R}\ H^{B}_{32} + {}^{R}W^{B}_{M/R}\ H^{B}_{33} \ . $$

THE LOCAL ALTITUDE H_M IS OBTAINED FROM

$$ H_M = H_M\ (0) + \int_0^T {}^{R}W^{H}_{M/R}\ DT \quad (11\text{-}23) $$

THE COMPONENTS ${}^{R}U^{H}_{M/R}$ AND ${}^{R}V^{H}_{M/R}$ ARE USED IN THE GENERATION OF LOCAL LATITUDE AND LONGITUDE RATES $\dot{\lambda}$ AND $\dot{L}$.

AN ORTHOGONAL TRANSFORMATION OF ${}^{R}\bar{V}^{H}_{M/R}$ TO THE RADAR REFERENCE FRAME $\bar{R}_I$ YIELDS

$$ {}^{R}U_{M/R} = {}^{R}U^{H}_{M/R}\ R^{H}_{11} + V^{H}_{M/R}\ R^{H}_{12} + W^{H}_{M/R}\ R^{H}_{13} $$

$$ {}^{R}V_{M/R} = {}^{R}U^{H}_{M/R}\ R^{H}_{21} + {}^{R}V^{H}_{M/R}\ R^{H}_{22} + {}^{R}W^{H}_{M/R}\ R^{H}_{23} \quad (11\text{-}24) $$

$$ {}^{R}W_{M/R} = {}^{R}U^{H}_{M/R}\ R^{H}_{31} + V^{H}_{M/R}\ R^{H}_{32} + {}^{R}W^{H}_{M/R}\ R^{H}_{33} \ . $$

THE VELOCITY OF THE MISSILE WITH RESPECT TO THE WIND NECESSARY FOR COMPUTING THE AERODYNAMIC FORCE IS OBTAINED FROM EQUATION (10-12) AS

$$\overline{V}_M = \overline{V}_{M/W} + {}^R\overline{V}_W + \overline{\omega}_E \times \overline{R}_M \tag{11-25}$$

AND BY EQUATION (11-20)

$$\overline{V}_M = {}^R\overline{V}_{M/R} + \overline{\omega}_E \times \overline{R}_M , \tag{11-26}$$

THEREFORE BY EQUATIONS (11-25) AND (11-26)

$$\overline{V}_{M/W} + {}^R\overline{V}_W = {}^R\overline{V}_{M/R} \tag{11-27}$$

THUS, IF THE VELOCITY OF THE WIND AS OBSERVED BY AN OBSERVER AT THE RADAR SITE IN THE LOCAL $\overline{H}_I$ FRAME IS TRANSFORMED INTO COMPONENTS IN THE $\overline{B}_I$ FRAME,

$$\begin{pmatrix} {}^RU^B_W \\ {}^RV^B_W \\ {}^RW^B_W \end{pmatrix} = \begin{pmatrix} B^H_{I_J} \end{pmatrix} \begin{pmatrix} {}^RU^H_W \\ {}^RV^H_W \\ {}^RW^H_W \end{pmatrix} \tag{11-28}$$

THE RECTANGULAR COORDINATES OF THE MISSILE WITH RESPECT TO THE RADAR SITE R IN THE RADAR REFERENCE FRAME $\overline{R}_I$ IS OBTAINED FROM THE EQUATIONS ABOVE AS

$$\begin{aligned} X^R_{M/R} &= X^R_{M/R}\,(0) + \int_0^T {}^RU_{M/R}\, DT \\ Y^R_{M/R} &= Y^R_{M/R}\,(0) + \int_0^T {}^RV_{M/R}\, DT \\ Z^R_{M/R} &= Z^R_{M/R}\,(0) + \int_0^T {}^RW_{M/R}\, DT \ . \end{aligned} \tag{11-29}$$

THE ORIENTATION OF THE $\overline{K}_{I_R}$ FRAME WITH RESPECT TO THE $\overline{R}_I$ FRAME IS GIVEN BY EQUATION (9-10) AS

$$\overline{K}_{I_R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\,(\lambda - \lambda^*) & s\,(\lambda - \lambda^*) \\ 0 & -s\,(\lambda - \lambda^*) & c\,(\lambda - \lambda^*) \end{bmatrix} \overline{R}_I \tag{11-30}$$

THE ORIENTATION OF THE $\overline{R}_I$ FRAME WITH RESPECT TO THE $\overline{H}$ FRAME IS GIVEN BY EQUATION (9-13) AS

$$\overline{R}_I = (R^H_{IJ})\ \overline{H}_I \qquad (11\text{-}31)$$

WHERE THE DIRECTION COSINES ARE GIVEN BY EQUATION (9-13) IN TERMS OF LOCAL LATITUDE AND LONGITUDE AND RADAR SITE LATITUDE AND LONGITUDE.

THE LOCAL LATITUDE AND LONGITUDE ARE OBTAINED FROM EQUATIONS (9-88) AS

$$\dot{\lambda} = \frac{{}^{R}V^{H}_{M}/R}{A\ F\ (\cos^2\lambda + 1) + A + H_M} \qquad (11\text{-}32)$$

$$\dot{L} = \frac{{}^{R}U^{H}}{(A + A\ F\ \sin^2\lambda + H_M)\ \cos\lambda}$$

THE ORTHOGONAL COMPONENTS OF $\overline{\omega}_H$ ARE GIVEN BY

$$\overline{\omega}_H = (\omega_E + \dot{L})\ \overline{E}_3 + \dot{\lambda}\overline{H}_1 \qquad (11\text{-}33)$$

WHERE BY EQUATION (9-8)

$$\overline{E}_3 = c\lambda\,\overline{H}_2 + s\lambda\,\overline{H}_3 \qquad (11\text{-}34)$$

AND CONSEQUENTLY,

$$\begin{aligned}\overline{\omega}_H &= \dot{\lambda}\overline{H}_1 + (\omega_E + \dot{L})\ c\lambda\,\overline{H}_2 + (\omega_E + \dot{L})\ s\lambda\,\overline{H}_3 \\ &= P_H\,\overline{H}_1 + Q_H\,\overline{H}_2 + R_H\,\overline{H}_3\end{aligned} \qquad (11\text{-}35)$$

DOTTING EQUATION (11-35) BY H_I RESPECTIVELY

$$\begin{aligned}P_H &= \dot{\lambda} \\ Q_H &= (\omega_E + \dot{L})\ c\lambda \\ R_H &= (\omega_E + \dot{L})\ s\lambda.\end{aligned} \qquad (11\text{-}36)$$

ROTATIONAL EQUATIONS OF MOTION ARE GIVEN BY

$$\dot{P}_B I^B_{11} + Q_B R_B (I^B_{33} - I^B_{22}) = L_A + L_T$$

$$\dot{Q}_B I^B_{22} + P_B R_B (I^B_{11} - I^B_{33}) = M_A + M_T \qquad (11\text{-}37)$$

$$\dot{R}_B I^B_{33} + P_B Q_B (I^B_{22} - I^B_{11}) = N_A + N_T,$$

WHERE L_T, M_T, AND N_T ARE GIVEN BY EQUATION (11-35).

SOLUTION OF MISSILE TRANSLATIONAL DYNAMICS IN BODY AXES $\frac{D}{DT} (\overline{V}_{M/W} + \overline{V}_W)$

THE TRANSLATIONAL ACCELERATION EQUATION OF THE MISSILE USED IN THIS SECTION IS OBTAINED FROM EQUATION (10-32) AS:

$$\overline{A}_M = \frac{{}^B\partial \overline{V}_{M/W}}{\partial T} + \overline{\omega}_B \times \overline{V}_{M/W} + \frac{{}^G\partial \overline{V}_W}{\partial T} + (\overline{\omega}_G + \overline{\omega}_E) \times {}^E\overline{V}_W + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) . \qquad (11\text{-}39)$$

WRITING THE INERTIAL ACCELERATION OF THE MISSILE OF EQUATION (11-39) IN THE TRANSLATIONAL DYNAMICS EXPRESSION $M\overline{A}_M = \overline{F}_T + \overline{F}_A + \overline{F}_{NG}$. (11-40)

IF $\overline{F}_T$ AND $\overline{F}_G$ ARE EXPRESSED IN TERMS OF COMPONENTS ALONG THE BODY AXES $\overline{B}_I$ THEN, $\overline{F}_A = F^B_{AI} \overline{B}_I$ AND, $\overline{F}_T = F^B_{TI} \overline{B}_I$ (11-41)

THUS, WRITING EQUATIONS (11-40) IN TERMS OF (11-39) AND (11-41)

$$M(T) \left\{ \frac{{}^G\partial {}^E\overline{V}_W}{\partial T} + (\overline{\omega}_G + \overline{\omega}_E) \times {}^E\overline{V}_W + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) + \frac{{}^B\partial \overline{V}_{M/W}}{\partial T} + \overline{\omega}_B \times \overline{V}_{M/W} \right\} = \overline{F}_A + \overline{F}_G + \overline{F}_T , \qquad (11\text{-}42)$$

WHICH UPON TRANSPOSING THE TERM $\overline{\omega}_E \times (\overline{\omega}_E \times R_M)$ TO THE RIGHT HANDED SIDE,

$$M\ (T) \left\{ \frac{{}^{G}\partial\, {}^{E}\overline{V}_W}{\partial\ T} + (\overline{\omega}_G + \overline{\omega}_E) \times {}^{E}\overline{V}_W + \frac{{}^{B}\partial\, {}^{E}\overline{V}_{M/W}}{\partial T} + \overline{\omega}_B \times \overline{V}_{M/W} \right\}$$

$$= \overline{F}_A + \overline{F}_T + (\overline{F}_G - M\,\overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M)) \tag{11-43}$$

THE EXPRESSION $\overline{F}_G - M \left\{ (\overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) \right\}$ WHICH IS THE RESULTANT FORCE DUE TO THE NEWTONIAN MASS ATTRACTION GRAVITATIONAL FORCE AND THE CENTRIFUGAL TERM DUE TO EARTH ROTATION IS CALLED THE "EFFECTIVE GRAVITY" AND IS IN THE DIRECTION OF A "PLUMB-BOB", THUS THE "EFFECTIVE GRAVITATIONAL" TERM IS

$$\overline{F}_G \ -M\,\overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) = + F_{G_E}\ \overline{G}_3 \tag{11-44}$$

THE VELOCITY OF THE WIND WITH RESPECT TO THE EARTH IS QUITE MEANINGFUL TO AN INDIVIDUAL WHEN SPECIFIED IN TERMS OF EASTERLY, NORTHERLY AND VERTICAL COMPONENTS, THUS ASSUMING WINDS SPECIFIED IN THE $\overline{H}_I$ FRAME.

$${}^{E}\overline{V}_W = {}^{E}U^{H}_{I_W}\ \overline{H}_I \ = \ {}^{E}U^{G}_{I_W}\ \ \overline{G}_I \tag{11-45}$$

AND

$$\frac{{}^{H}\partial\, {}^{E}\overline{V}_W}{\partial\ T} = {}^{H}\dot{U}^{H}_{I_W}\ \overline{H}_I \ = \ {}^{G}\dot{U}^{G}_{I_W}\ \ \overline{G}_I \tag{11-46}$$

THE TERM $\overline{\omega}_E \times {}^{E}\overline{V}_W$ OF EQUATION (12-5) REQUIRES $\overline{\omega}_E$ COMPONENTS IN THE $\overline{G}_I$ FRAME, THUS IF

$$\overline{G}_I = (G^{E}_{IJ})\ \ \overline{E}_I \tag{11-47}$$

THEN

$$\overline{E}_3 = G^{E}_{13}\ \overline{G}_1 \ + \ G^{E}_{23}\ \overline{G}_2 \ + \ G^{E}_{33}\ \overline{G}_3\ . \tag{11-48}$$

WRITING EQUATION (11-43) IN COMPONENTS ALONG BODY AXES $\overline{B}_I$ AND $\overline{G}_I$ AXES, BY EQUATIONS (11-45), (11-46) AND (11-48).

$$M(T)\left\{{}^{GE}\dot{U}_W \bar{G}_1 + {}^{GE}\dot{V}_W \bar{G}_2 + {}^{GE}\dot{W}_W \bar{G}_3 + \begin{vmatrix} \bar{G}_1 & \bar{G}_2 & \bar{G}_3 \\ P_G + {}_E\, G^E_{13} & Q_G + {}_E\, G^E_{23} & R_G + {}_E\, G^E_{33} \\ {}^E U^G_W & {}^E V^G_W & {}^E W^G_W \end{vmatrix}\right.$$

$$+ {}^B\dot{U}_{M/W}\, B_1 + {}^B\dot{B}_{M/W}\, B_2 + {}^B\dot{W}_{M/W}\, B_3 + \begin{vmatrix} \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ P_B & Q_B & R_B \\ U^B_{M/W} & V^B_{M/W} & W^B_{M/W} \end{vmatrix}$$

$$= F^B_{A_1} B_1 + F^B_{T_1} B_1 + F_{GE}\, G_3. \tag{11-49}$$

EXPANDING THE DETERMINANTS OF THE ABOVE EQUATION

$$M(T)\left\{{}^{GE}\dot{U}_W + {}^E W^G_W (Q_G + \omega_E\, G^E_{23}) - {}^E V^G_W (R_G + \omega_E\, G^E_{33})\right\} G_1 + \left\{{}^{GE}\dot{U}_W - {}^E W^G_W (P_G + \omega_E\, G^E_{13}) + {}^E U^G_W (R_G + \omega_E\, G^E_{33})\right\} G_2 + \left\{{}^{GE}\dot{W}_W + {}^E V^G_2 (P_G + \omega_E\, G^E_{13}) - {}^E U^G_W (Q_G + \omega_E\, G^E_{23})\right\} G_3$$

$$+ \left\{{}^B\dot{U}_{M/W} + Q_B W^B_{M/W} R_B\right\} B_1 + \left\{{}^B\dot{V}_{M/W} - W^B_{M/W} P_B + U^B_{M/W} R_B\right\} \bar{B}_2 + \left\{{}^B\dot{W}_{M/W} - U^B_{M/W} Q_B + V^B_{M/W} P_B\right\} \bar{B}_3 = F^B_{A_1} \bar{B}_1 + F^B_{A_2} \bar{B}_2 + F^B_{A_3} \bar{B}_3 + F^B_{T_1} \bar{B}_1 + F^B_{T_2} \bar{B}_2 + F^B_{T_3} \bar{B}_3 + F_{GE}\, \bar{G}_3 \tag{11-50}$$

THE DIRECTION COSINE MATRIX BETWEEN THE BODY FRAME AND THE LOCAL GROUND FRAME $\bar{G}_I$ IS $\bar{B}_I = (B^G_{IJ})\, \bar{G}_I$ (11-51)

DOTTING EQUATION (11-50) BY B_I RESPECTIVELY AND UTILIZING EQUATION (11-51)

$$M\left\{({}^B\dot{U}_{M/W} + Q_B W^B_{M/W} - V^B_{M/W} R_B) + \left[{}^{GE}\dot{U}_W + {}^E W^G_W (Q_G + \omega_E\, G^E_{23}) - {}^E V^G_W (R_G + \omega_E\, G^E_{33})\right] B_{11} + \left[{}^{GE}\dot{V}_W - {}^E W^G_W (P_G + \omega_E G^E_{13}) + {}^E U^G_W (R_G + {}_E\, G^E_{33})\right] B_{12} + \left[{}^{GE}W_W + {}^E V^G_W (P_G + \omega_E\, G^E_{13}) - U^G_W (Q_G + \omega_E\, G^E_{23})\right] B^G_{13}\right\} = F^B_{A_1} + F^B_{T_1} + F_{G_E}\, B^G_{13}$$

$$
\begin{aligned}
&M\Big\{\left({}^{B}\dot{V}_{M/W} - W^{B}_{M/W}\ P_B + R_B\ U^{B}_{M/W}\right) + \Big[{}^{GE}\dot{U}_W + {}^{E}W^{G}_{W}\ \left(Q_G + \omega_E\ G^{E}_{23}\right) - {}^{E}V^{G}_{W} \\
&\left(R_G + \omega_E\ G^{E}_{33}\right)\Big]\ B^{G}_{21} + \Big[{}^{GE}\dot{V}_W - {}^{E}W^{G}_{W}\ \left(P_G + \omega_E\ G^{E}_{13}\right) + {}^{E}U^{G}_{W}\ \left(R_G + {}_{E}\ G^{E}_{33}\right)\Big]\ B_{22} \\
&+ \Big[{}^{GE}W_W + {}^{E}V^{G}_{W}\ \left(P_G + \omega_E\ G^{E}_{13}\right) - {}^{E}U^{G}_{W}\ \left(Q_G + \omega_E\ G^{E}_{23}\right)\Big]\ B^{G}_{23}\Big\} = F^{B}_{A2} + F^{B}_{T2} + F_{GE}\ B^{G}_{23}
\end{aligned}
$$

$$
\begin{aligned}
&M\Big\{\left({}^{B}\dot{W}_{M/W} - Q_B\ U^{B}_{M/W} + P_B\ V^{B}_{M/W}\right) + \Big[{}^{GE}\dot{U}_W + {}^{E}W^{G}_{W}\ \left(Q_G + \bar{\omega}_E\ G^{E}_{23}\right) - {}^{E}V^{G}_{W}\ \left(R_G + \bar{\omega}_E\ G^{E}_{31}\right) B^{G}_{31} \\
&+ \Big[{}^{GE}\dot{V}_W - {}^{E}W^{G}_{W}\ \left(P_G + \omega_E\ G^{E}_{13}\right) + {}^{E}U^{G}_{W}\ \left(R_G + \omega_E\ G^{E}_{33}\right)\Big]\ B^{E}_{32} + \Big[{}^{GE}\dot{W}_W + V^{G}_{W}\ \left(P_G + \omega_E\ G^{E}_{13}\right) \\
&- {}^{E}U^{G}_{W}\ \left(Q_G + \omega_E\ G^{E}_{23}\right)\Big]\ B^{G}_{33}\Big\} = F^{B}_{A_3} + F^{B}_{T_3} + F_{GE}\ B^{G}_{33}. \qquad (11\text{-}52)
\end{aligned}
$$

THE DIRECTION COSINES B^{G}_{IJ} OF EQUATION (11-52) ARE GIVEN BY EQUATION (11-51).

THE VELOCITY OF THE MISSILE WITH RESPECT TO THE WIND IS OBTAINED FROM THE SOLUTION OF EQUATION (11-52). BY EQUATION (10-12) AND EQUATION (10-16).

$$
\overline{V}_M = {}^{E}\overline{V}_M + \overline{\omega}_E \times \overline{R}_M = \overline{V}_{M/W} + \overline{V}_W = V_{M/W} + {}^{E}\overline{V}_W + \overline{\omega}_E \times \overline{R}_M \text{ HENCE } {}^{E}\overline{V}_M = \overline{V}_{M/W} + {}^{E}\overline{V}_W\ . \qquad (11\text{-}53)
$$

SINCE THE RADAR FRAME IS FIXED TO THE EARTH, THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH IS EQUAL TO THE VELOCITY OF THE MISSILE WITH RESPECT TO THE RADAR FRAME.

$$
{}^{E}\overline{V}_M = {}^{E}U^{R}_{M}\ \overline{R}_1 + {}^{E}V^{R}_{M}\ \overline{R}_2 + {}^{E}W^{R}_{M}\ \overline{R}_3\ . \qquad (11\text{-}54)
$$

FURTHERMORE, UTILIZING THE DIRECTION COSINE MATRIX BETWEEN THE RADAR FRAME AND THE BODY FRAME $\overline{B}_I = \left(B^{R}_{IJ}\right)\overline{R}_I$, ONE MAY OBTAIN

$$
\overline{V}_{M/W} = U^{B}_{M/W}\ \overline{B}_1 + V^{B}_{M/W}\ \overline{B}_2 + W^{B}_{M/W}\ \overline{B}_3 = U^{R}_{M/W}\ \overline{R}_1 + V^{R}_{M/W}\ \overline{R}_2 + W^{R}_{M/W}\ \overline{R}_3\ , \qquad (11\text{-}55)
$$

DOTTING EQUATION (11-55) BY $\overline{R}_I$ RESPECTIVELY

$$U^R_{M/W} = U^B_{M/W} \; B^R_{11} + V^B_{M/W} \; B^R_{21} + W^B_{M/W} \; B^R_{31}$$

$$V^R_{M/W} = U^B_{M/W} \; B^R_{12} + V^B_{M/W} \; B^R_{22} + W^B_{M/W} \; B^R_{32}$$

$$W^R_{M/W} = U^B_{M/W} \; B^R_{13} + V^B_{M/W} \; B^R_{23} + W^B_{M/W} \; B^R_{33} \; . \qquad (11\text{-}56)$$

EXPRESSING THE VELOCITY OF THE MISSILE WITH RESPECT TO THE RADAR FRAME AS GIVEN BY EQUATION (11-53), (11-54), AND (11-56).

$$^RU_{M/R} \, \overline{R}_1 + {}^RV_{M/R} \, \overline{R}_2 + {}^RW_{M/R} \, \overline{R}_3 = U^R_{M/W} \, \overline{R}_1 + V_{M/W} \, \overline{R}_2$$

$$+ W^R_{M/R} \, \overline{R}_3 + {}^EU^G_W \, \overline{G}_1 + {}^EV^G_W \, \overline{G}_2 + {}^EW^G_W \, \overline{G}_3 \; . \qquad (11\text{-}57)$$

DOTTING EQUATION (11-57) BY $\overline{R}_I$ RESPECTIVELY AND DESIGNATING THE DIRECTION COSINE MATRIX BY $\overline{G}_I = (G^R_{IJ}) \, \overline{R}_I$,

$$(11\text{-}58)$$

ONE OBTAINS

$$U^R_{M/R} = U^R_{M/W} + ({}^EU^G_W \; G^R_{11} + {}^EV^G_W \; G^R_{21} + {}^EW^G_W \; G^R_{31})$$

$$V^R_{M/R} = V^R_{M/W} + ({}^EU^G_W \; G^R_{12} + {}^EV^G_W \; G^R_{22} + {}^EW^G_W \; G^R_{32})$$

$$W^R_{M/R} = W^R_{M/W} + ({}^EU^G_W \; G^R_{13} + {}^EV^G_W \; G^R_{23} + {}^EW^G_W \; G^R_{33}) . \qquad (11\text{-}59)$$

THE RADAR RECTANGULAR COORDINATES ARE:

$$X^R_{M/R} = X^R_{M/R} \; (0) + \int_0^T U^R_{M/R} \; DT$$

$$Y^R_{M/R} = Y^R_{M/R} \; (0) + \int_0^T V^R_{M/R} \; DT$$

$$Z^R_{M/R} = Z^R_{M/R} \; (0) + \int_0^T W^R_{M/R} \; DT \; . \qquad (11\text{-}60)$$

BLOCK DIAGRAM IS SHOWN IN FIGURE **11-1.**

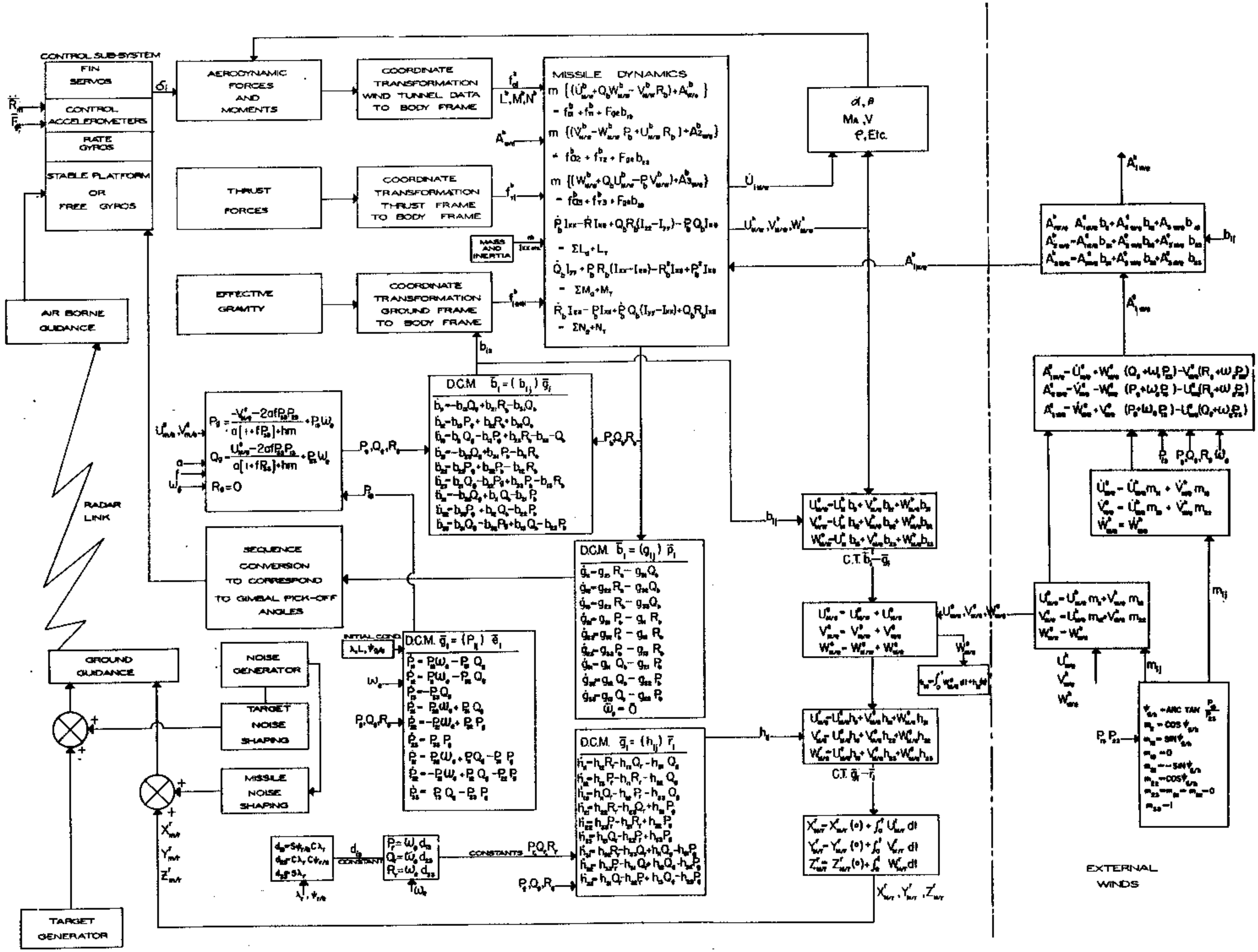


FIG. II-1. MMAND GUIDANCE MISSILE SYSTEM OVER SPHEROIDAL E/ 1 WITH MISSILE DYNAMICS SOLVED IN BODY AXES $\left(\frac{D}{DT}\right.$ ... V_{MIW}

SOLUTION OF MISSILE DYNAMICS IN GROUND FIXED TRACKER AXES $\frac{D^2}{DT^2}\ (\overline{R}_R + \overline{R}_{M/R})$

THE MISSILE TRANSLATIONAL ACCELERATION EQUATIONS SOLVED IN A GROUND FIXED REFERENCE FRAME ON THE SURFACE OF THE EARTH (E.G. A TRACKER-RADAR REFERENCE FRAME) ARE AS FOLLOWS:

THE POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE INERTIAL ORIGIN IS EQUAL TO THE POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE RADAR SITE PLUS THE POSITION VECTOR OF THE RADAR SITE WITH RESPECT TO THE INERTIAL ORIGIN AS SHOWN IN FIGURE 11-2, HENCE $\overline{R}_M = \overline{R}_R + \overline{R}_{M/R}$

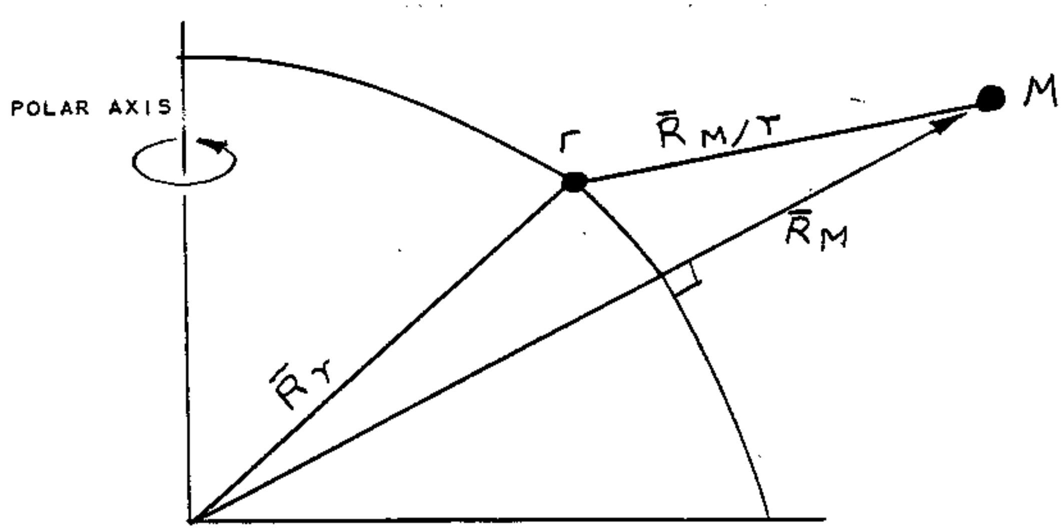


FIG. 11-2. RADAR ORIGIN ON SPHEROIDAL EARTH.

IF THE COORDINATES OF $\overline{R}_R$ ARE EXPRESSED IN THE EARTH FRAME AND THE COORDINATES OF $\overline{R}_{M/R}$ ARE EXPRESSED IN THE RADAR FRAME ONE HAS:

$$\overline{R}_R = X_R^E\ \overline{E}_1 + Y_R^E\ \overline{E}_2 + Z_R^E\ \overline{E}_3 \text{ AND } \overline{R}_{M/R} = X_{M/R}^R\ \overline{R}_1 + Y_{M/R}^R\ \overline{R}_2 + Z_{M/R}^R\ R_3. \tag{11-63}$$

TAKING THE TIME DERIVATIVE OF EQUATION (11-63)

$$\frac{D\,\bar{R}_M}{DT} = \bar{V}_M = \frac{{}^{E}\partial\,\bar{R}^{E}}{\partial T} + \bar{\omega}_E \times \bar{R}_R + {}^{R}\bar{R}_{M/R} + \bar{\omega}_R \times \bar{R}_{M/R} \qquad (11\text{-}64)$$

SINCE THE RADAR ORIGIN IS FIXED TO THE EARTH ITS COORDINATES DO NOT CHANGE IN THE EARTH FRAME, HENCE $\dfrac{{}^{R}\partial\,\bar{R}^{E}_{R}}{\partial T} = \bar{0}$ AND $\bar{\omega}_R = \bar{\omega}_E$, HENCE

$$\bar{V}_M = \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T} + \bar{\omega}_E \times \bar{R}_M . \qquad (11\text{-}65)$$

TAKING THE DERIVATIVE OF THE VELOCITY VECTOR OF EQUATION (11-65)

$$\bar{A}_M = \frac{D\,\bar{V}_M}{DT} = \frac{{}^{RR}\partial^2\bar{R}_{M/R}}{\partial T^2} + \bar{\omega}_R \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T} + \bar{\omega}_E \times \frac{D\,\bar{R}_M}{DT} \qquad (11\text{-}66)$$

AND BY EQUATION (11-65)

$$\bar{A}_M = \frac{{}^{RR}\partial^2\bar{R}_{M/R}}{\partial T^2} + \bar{\omega}_R \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T} + \bar{\omega}_E \times \left[\frac{{}^{R}\partial\,\bar{R}_M}{\partial T} + \bar{\omega}_E \times \bar{R}_M\right]$$

OR $$\bar{A}_M = \frac{{}^{RR}\partial^2\bar{R}_{M/R}}{T^2} + 2\bar{\omega}_E \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T} + \bar{\omega}_E \times (\bar{\omega}_E \times \bar{R}_M) \qquad (11\text{-}67)$$

UTILIZING EQUATION (11-67) IN EQUATION (11-40)

$$M\left\{\frac{{}^{RR}\partial^2\bar{R}_{M/R}}{\partial T^2} + 2\bar{\omega}_E \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T} + \bar{\omega}_E \times (\bar{\omega}_E \times \bar{R}_M)\right\} = \bar{F}_A + \bar{F}_T + \bar{F}_G \qquad (11\text{-}68)$$

SUBTRACTING THE CENTRIPETAL ACCELERATION TERM

$$M\left\{\frac{{}^{RR}\partial^2\bar{R}_{M/R}}{\partial T^2} + 2\bar{\omega}_E \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T}\right\} = \bar{F}_A + \bar{F}_T + \bar{F}_{NG} - \bar{\omega}_E \times (\omega_E \times \bar{R}_M) \qquad (11\text{-}69)$$

BY EQUATION (11-44)

$$M\left\{\frac{{}^{RR}\partial^2\bar{R}_{M/R}}{\partial T^2} + 2\bar{\omega}_E \times \frac{{}^{R}\partial\,\bar{R}_{M/R}}{\partial T}\right\} = \bar{F}_A + \bar{F}_T + F_{GE}\ \bar{G}_3 \qquad (11\text{-}70)$$

THE DIRECTION COSINE MATRIX ORIENTING THE RADAR FRAME $\overline{R}_1$ WITH RESPECT TO THE EARTH FRAME $\overline{E}_1$ IS GIVEN BY:

$$\overline{R}_I = \left(R^E_{IJ}\right) \overline{E}_I \tag{11-71}$$

THUS,

$$\overline{E}_3 = R^E_{13}\,\overline{R}_1 + R^E_{23}\,\overline{R}_2 + R^E_{33}\,\overline{R}_3 \,. \tag{11-72}$$

UTILIZING EQUATION (11-72) IN (11-70) AND EXPRESSING THE VECTORS IN COMPONENT FORM:

$$M\left\{\ddot{X}^R_{M/R}\,R_1 + \ddot{Y}^R_{M/R}\,R_2 + \ddot{Z}^R_{M/R}\,\overline{R}_3 + 2\omega_E \begin{vmatrix} \overline{R}_1 & \overline{R}_2 & \overline{R}_3 \\ R^E_{13} & R^E_{23} & R^E_{33} \\ X^R_{M/R} & Y^R_{M/R} & Z^R_{M/R} \end{vmatrix}\right.$$

$$= \left(F^B_{A_I} + F^B_{T_I}\right)\overline{B}_I + F_{GE}\,\overline{H}_3 \,. \tag{11-73}$$

THE DIRECTION COSINE MATRIX ORIENTING THE BODY FRAME WITH RESPECT TO THE RADAR FRAME IS:

$$\overline{B}_I = \left(B^R_{IJ}\right) \overline{R}_I \,. \tag{11-74}$$

THE DIRECTION COSINE MATRIX OF EQUATION (11-74) ORIENTING THE MISSILE BODY AXES WITH RESPECT TO THE GROUND FIXED RADAR FRAME R_I MAY BE OBTAINED IN ANY OF THE METHODS DISCUSSED IN SECTION 2.

THE ORIENTATION OF THE LOCAL EAST, NORTH, VERTICAL FRAME $\overline{H}_I$ WITH RESPECT TO THE RADAR FRAME $\overline{R}_I$ IS GIVEN BY EQUATION (9-13) AS:

$$\left(\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} C\,(L-L_R) & S\lambda_R\,S\,(L-L_R & -C\lambda_R\,S\,(L-L_R \\ -S\lambda S(L-L_R) & S\lambda S\lambda_R\,C(L-L_R) + C\lambda C\lambda_R & -S\lambda C\lambda_R\,C(L-L_R) + C\lambda C\lambda_R \\ C\lambda S(L-L_R) & -C\lambda S\lambda_R\,C(L-L_R) + S\lambda C\lambda_R & C\lambda C\lambda_R\,C(L-L_R) + S\lambda S\lambda_R \end{pmatrix} R_I\right)$$

$$\overline{H}_I = \left(H^R_{IJ}\right)\ \overline{R}_I \tag{11-75}$$

THE LATITUDE AND LONGITUDE OF THE MISSILE CAN BE COMPUTED FROM EQUATION (9-88) AS:

$$\dot{\lambda} = \frac{{}^{R}V_M^H}{AF\ (\cos^2 \lambda + 1) + A + H_M}$$

$$\dot{L} = \frac{{}^{R}U_M^H}{(A + AF\ \sin^2 \lambda + H_M)\ \cos \lambda} \qquad (11\text{-}76)$$

THE COMPONENTS OF VELOCITY OF THE MISSILE ${}^{R}U_M^H$, ${}^{R}V_M^H$, ${}^{R}W_M^H$ MAY BE OBTAINED FROM ${}^{R}U_M^H\ \bar{H}_1 + {}^{R}V_M^H\ \bar{H}_2 + {}^{R}W_M^H\ \bar{H}_3 = {}^{R}U_M^R\ \bar{R}_1 + {}^{R}V_M^R\ \bar{R}_2 + {}^{R}W_M^R\ \bar{R}_3$

(11-77)

DOTTING EQUATION (11-77) BY $\bar{H}_I$ RESPECTIVELY

$$ {}^{R}U_M^H = {}^{R}U_M\ H_{11}^R + {}^{R}V_M\ H_{12}^R + {}^{R}W_M\ H_{13}$$

$$ {}^{R}V_M^H = {}^{R}U_M\ H_{21}^R + {}^{R}V_M\ H_{22}^R + {}^{R}W_M\ H_{23}^R$$

$$ {}^{R}W_M^H = {}^{R}U_M\ H_{31}^R + {}^{R}V_M\ H_{32}^R + {}^{R}W_M\ H_{33}^R \qquad (11\text{-}78)$$

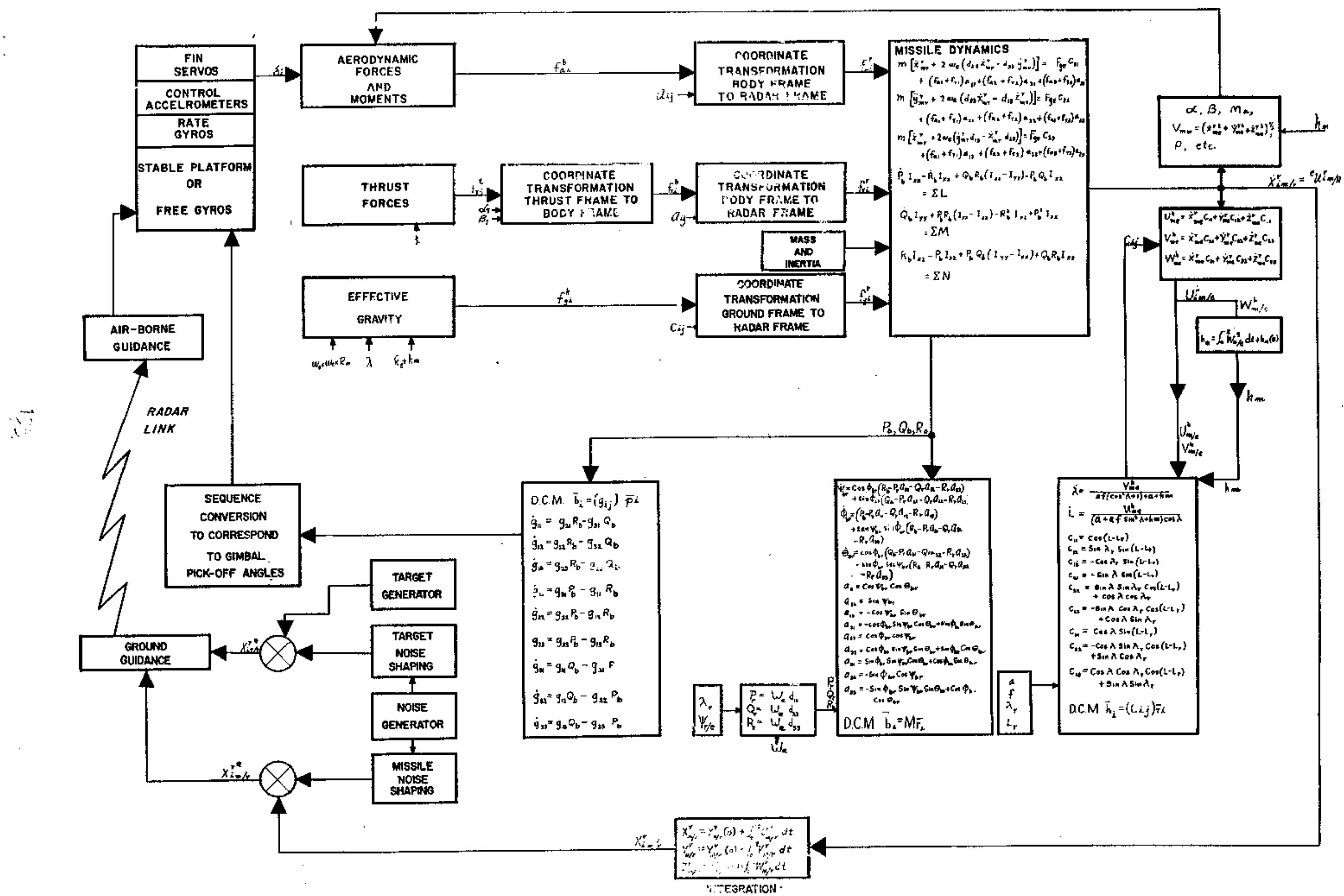
FINALLY THE SCALAR EQUATIONS OBTAINED FROM EQUATION (11-73) UTILIZING EQUATION (11-74) AND EQUATION (11-75), FIGURE 11-2A.

$$M\left[{}^{RR}\ddot{X}_{M/R} + 2\omega_E(H_{23}^R\ {}^{R}\dot{Z}_{M/R} - H_{33}^R\ {}^{R}\dot{Y}_{M/R})\right] = (F_{A_1} + F_{T_1})\ \bar{B}_1 \cdot \bar{R}_1 + F_{GE}\ \bar{H}_3 \cdot \bar{R}_1$$

$$M\left[{}^{RR}\ddot{Y}_{M/R} + 2\omega_E\ (H_{33}^R\ {}^{R}\dot{X}_{M/R} - {}^{R}\dot{Z}_{M/R}\ H_{13}^R)\right] = (F_{A_1} + F_{T_1})\ \bar{B}_1 \cdot \bar{R}_2 + F_{GE}\ \bar{H}_3 \cdot \bar{R}_2$$

$$M\left[{}^{RR}\ddot{Z}_{M/R} + 2\omega_E\ (H_{13}^R\ {}^{R}\dot{Y}_{M/R} - {}^{R}\dot{X}_{M/R}\ H_{23}^R)\right] = (F_{A_1} + F_{T_1})\ \bar{B}_1 \cdot \bar{R}_3 + F_{GE}\ \bar{H}_3 \cdot \bar{R}_3\ .$$

(11-79)


FIN SERVOS
CONTROL ACCELROMETERS
RATE GYROS
STABLE PLATFORM OR FREE GYROS
AERODYNAMIC FORCES AND MOMENTS
COORDINATE TRANSFORMATION BODY FRAME TO RADAR FRAME
MISSILE DYNAMICS
THRUST FORCES
COORDINATE TRANSFORMATION THRUST FRAME TO BODY FRAME
MASS AND INERTIA
EFFECTIVE GRAVITY
COORDINATE TRANSFORMATION GROUND FRAME TO RADAR FRAME
AIR-BORNE GUIDANCE
RADAR LINK
SEQUENCE CONVERSION TO CORRESPOND TO GIMBAL PICK-OFF ANGLES
TARGET GENERATOR
TARGET NOISE SHAPING
NOISE GENERATOR
MISSILE NOISE SHAPING
GROUND GUIDANCE
INTEGRATION

THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH IN THE RADAR FRAME IS GIVEN BY:

$$
\begin{aligned}
{}^{R}U_{M/R} &= \dot{X}^{R}_{M/R} \\
{}^{R}V_{M/R} &= \dot{Y}^{R}_{M/R} \\
{}^{R}W_{M/R} &= \dot{Z}^{R}_{M/R} \ ,
\end{aligned}
\qquad (11\text{-}80)
$$

AND RADAR RECTANGULAR COORDINATES ARE GIVEN BY:

$$
\begin{aligned}
X^{R}_{M/R} &= X^{R}_{M/R}\ (0) + \int_{0}^{T} \dot{X}^{R}_{M/R}\ DT \\
Y^{R}_{M/R} &= Y_{M/R}\ (0) + \int_{0}^{T} \dot{Y}^{R}_{M/R}\ DT \\
Z^{R}_{M/R} &= Z^{R}_{M/R}\ (0) + \int_{0}^{T} \dot{Z}^{R}_{M/R}\ DT
\end{aligned}
\qquad (11\text{-}81)
$$

THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH IS GIVEN BY

$$
V_{M/R} = \left[{}^{R}\dot{X}^{2}_{M/R} + {}^{R}\dot{Y}^{2}_{M/R} + {}^{R}\dot{Z}^{2}_{M/R} \right]^{1/2} \qquad (11\text{-}82)
$$

AND FOR A STUDY WITH NO EXTERNAL WINDS THIS IS THE VELOCITY NEEDED TO GENERATE THE AERODYNAMIC FORCES, I.E.

$$
\overline{V}_{M/W} = {}^{E}\overline{V}_{M/O} \ . \qquad (11\text{-}83)
$$

SOLUTION IN INERTIAL FRAME

THE TRANSLATIONAL ACCELERATION EQUATIONS ARE OFTEN SOLVED IN AN INERTIAL FRAME OF REFERENCE, CONSEQUENTLY THIS SECTION WILL DERIVE A SET OF EQUATIONS IN SUCH A FRAME.

THE POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE INERTIAL ORIGIN OF FIGURE 11-2 EXPRESSED IN INERTIAL AXES $\overline{I}_I$ IS -

$$\overline{R}_M = X_M^I \overline{I}_1 + Y_M^I \overline{I}_2 + Z_M^I \overline{I}_3 \qquad (11\text{-}84)$$

AND TAKING THE TIME DERIVATIVE

$$\overline{V}_M = \frac{{}^I\partial \overline{R}_M}{\partial T} + \overline{\omega}_I \times \overline{R}_M$$

BUT $\overline{\omega}_I = \overline{0}$, THEREFORE, $\overline{V}_M = \dot{X}_M^I \overline{I}_1 + \dot{Y}_M^I \overline{I}_2 + \dot{Z}_M^I \overline{I}_3$. (11-85)

THE INERTIAL ACCELERATION IS GIVEN BY:

$$A_M = \ddot{X}_M \overline{I}_1 + \ddot{Y}_M \overline{I}_2 + \ddot{Z}_M \overline{I}_3 \ . \qquad (11\text{-}86)$$

UTILIZING EQUATION (11-86) IN EQUATION (11-3)

$$M \left[\ddot{X}_M I_1 + \ddot{Y}_M I_2 + \ddot{Z}_M I_3\right] = F_{A_I}^B \overline{B}_I + F_{T_I}^B \overline{B}_I + \overline{F}_G$$

DOTTING EQUATION (11-86) BY I_I RESPECTIVELY

$$M \ddot{X}_M^I = (F_{A_I}^B + F_{T_I}^B)\ \overline{B}_I \cdot \overline{I}_1 + \overline{F}_G \cdot \overline{I}_1$$

$$M \ddot{Y}_M^I = (F_{A_I}^B + F_{T_I}^B)\ \overline{B}_I \cdot \overline{I}_2 + \overline{F}_G \cdot \overline{I}_2$$

$$M \ddot{Z}_M^I = (F_{A_I}^B + F_{T_I}^B)\ \overline{B}_I \cdot \overline{I}_3 + \overline{F}_G \cdot \overline{I}_3 \qquad (11\text{-}88)$$

THE ADVANTAGE OF THIS SOLUTION LIES IN THE FACT THAT THE THREE COMPONENTS OF INERTIAL ANGULAR VELOCITY OF THE INERTIAL FRAME P_I, Q_I, R_I ARE ZERO, THUS SIMPLIFYING THE COMPUTATIONS TO OBTAIN THE DIRECTION COSINES.

SOLUTION OF TRANSLATIONAL EQUATIONS IN RELATIVE VELOCITY VECTOR FRAME

THE TRANSLATIONAL ACCELERATION EQUATIONS ARE SOLVED IN THE $\overline{T}_I^*$ REFERENCE FRAME. THIS FRAME (FIG. 11-3) HAS THE UNIT $\overline{T}_1^*$ VECTOR IN THE DIRECTION OF THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH, $\overline{T}_3^*$ LIES IN THE PLANE CONTAINING THE LOCAL VERTICAL $\overline{G}_3$ AND THE T_1^* VECTOR,

AND T_2^* IS NORMAL TO THE INSTANTANEOUS "RELATIVE VELOCITY VECTOR - VERTICAL PLANE", DESCRIBED ABOVE.

THE ORIENTATION OF THE T_I^* FRAME WITH RESPECT TO THE LOCAL PLUMB-BOB FRAME, G_I IS GIVEN BY A PITCH AND A YAW MATRIX:

$$\bar{T}_I^* = M_Y \left(\psi_{T^*/G}\right) M_P \left(\theta_{T^*/G}\right) \bar{G}_I \tag{11-89}$$

OR,

$$\begin{pmatrix} \bar{T}_1^* \\ \bar{T}_2^* \\ \bar{T}_3^* \end{pmatrix} = \begin{pmatrix} C\,\theta_{T^*/G}\; C\psi_{T^*/G} & C\,\theta_{T^*/G}\; S\psi_{T^*/G} & -S\,\theta_{T^*/G} \\ -S\psi_{T^*/G} & C\,\psi_{T^*/G} & 0 \\ C\psi_{T^*/G}\; S\,\theta_{T^*/G} & S\,\theta_{T^*/G}\; S\,\psi_{T^*/G} & C\,\theta_{T^*/G} \end{pmatrix} \begin{pmatrix} \bar{G}_1 \\ \bar{G}_2 \\ \bar{G}_3 \end{pmatrix}$$

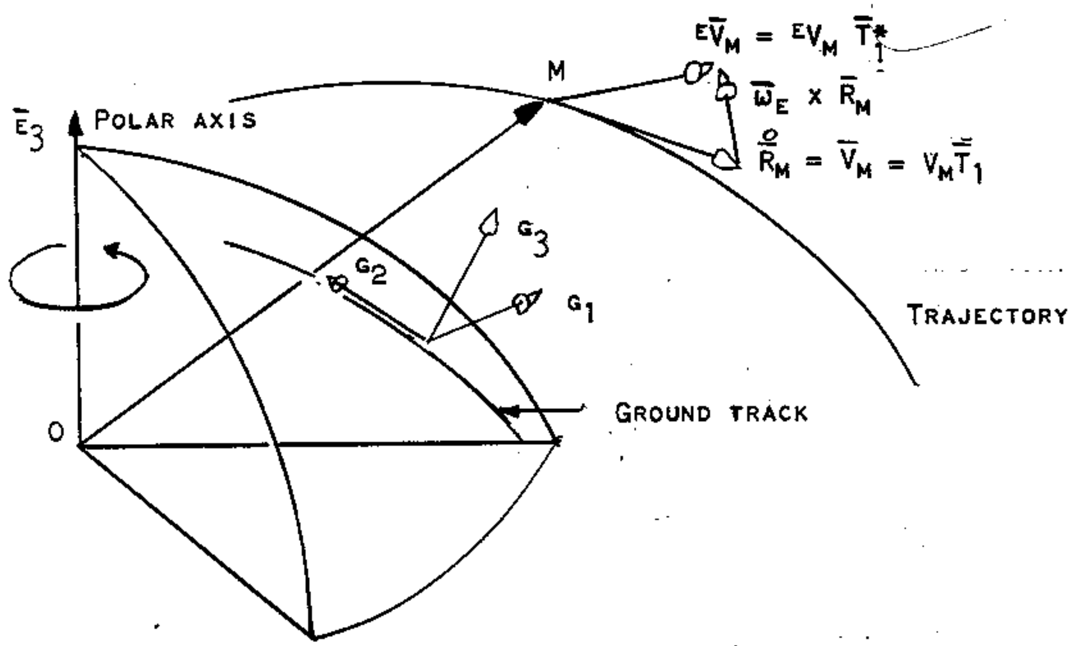


FIG. 11-3. VELOCITY OF MISSILE WITH RESPECT TO GROUND VECTOR.

CONSIDER THE POSITION VECTOR OF THE MISSILE CENTER OF MASS WITH RESPECT TO AN INERTIAL ORIGIN O AT THE CENTER OF AN OBLATE SPHEROIDAL EARTH AS SHOWN IN FIGURE 11-3.

THE TIME DERIVATIVE OF $\overline{R}_M$ IS THE VELOCITY OF POINT M WITH RESPECT TO POINT O, THUS IF $\overline{R}_M$ IS EXPRESSED IN THE EARTH FRAME $\overline{E}_i$,

$$\overline{R}_M = X^E_{iM} \quad \overline{E}_i \qquad (i = 1, 2, 3) \tag{11-90}$$

AND,

$$\dot{\overline{R}}_M = \overline{V}_M = \frac{{}^E\partial \overline{R}_M}{\partial T} + \overline{\omega}_E \times \overline{R}_M \tag{11-91}$$

WHERE,

$$\frac{{}^E\partial \overline{R}_M}{\partial T} = \dot{X}^E_{iM} \quad \overline{E}_i = {}^E\overline{V}_M \; . \tag{11-92}$$

IF THE VELOCITY OF THE MISSILE WITH RESPECT TO THE EARTH IS EXPRESSED IN THE $\overline{T}^*_i$ FRAME AND THE $\overline{E}_i$ FRAME,

$${}^E\overline{V}_M = U^E_{iM/E} \quad \overline{E}_i = {}^EV^{T*}_{M_i} \quad \overline{T}^*_i \; . \tag{11-93}$$

TAKING THE TIME DERIVATIVE OF EQUATION (11-93)

$$\frac{D\ {}^E\overline{V}_M}{DT} = \frac{{}^E\partial\ {}^E\overline{V}_M}{\partial T} + \overline{\omega}_E \times {}^E\overline{V}_M = \frac{{}^{T*}\partial\ {}^EV^{T*}_{M_i}}{\partial T} + \overline{\omega}_{T*} \times {}^E\overline{V}_M \tag{11-94}$$

AND SOLVING FOR $\dfrac{{}^E\partial\ {}^E\overline{V}_M}{\partial T}$, ONE OBTAINS:

$$\frac{{}^E\partial\ {}^EV_M}{\partial T} = \frac{{}^{T*}\partial\ {}^EV^{T*}_M}{\partial T} + (\overline{\omega}_{T*} - \overline{\omega}_E) \times {}^E\overline{V}_M \; , \tag{11-95}$$

WHERE THE EXPRESSION $\dfrac{{}^E\partial\ {}^E\overline{V}_M}{\partial T}$ MEANS

$$\frac{{}^E\partial\ {}^E\overline{V}_M}{\partial T} = {}^{EE}\dot{U}_{iM} \quad \overline{E}_i \quad (i = 1, 2, 3) \; . \tag{11-96}$$

THE INERTIAL ACCELERATION OF THE MISSILE IS GIVEN AS THE TIME DERIVATIVE OF THE VELOCITY VECTOR OF EQUATION (11-91).

$$\frac{D^2 \overline{R}_M}{DT^2} = \frac{D \; \dot{\overline{V}}_M}{DT} = \overline{A}_M = \frac{{}^{E}\partial \, {}^{E}\overline{V}_M}{\partial T} + \overline{\omega}_E \times {}^{E}\overline{V}_M$$

$$+ \overline{\omega}_E \times ({}^{E}\overline{V}_M + \overline{\omega}_E \times \overline{R}_M) \; . \qquad (11\text{-}97)$$

SUBSTITUTING EQUATION (11-95) INTO EQUATION (11-97):

$$\overline{A}_M = \frac{{}^{T*}\partial \, {}^{E}\overline{V}_M}{\partial T} + (\overline{\omega}_{T*} - \overline{\omega}_E) \times {}^{E}\overline{V}_M + 2 \, \overline{\omega}_E \times {}^{E}\overline{V}_M + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) \; . \quad (11\text{-}98)$$

THE ANGULAR VELOCITY OF THE $\overline{T}^*_1$ FRAME WITH RESPECT TO THE EARTH IS:

$$\overline{\omega}_{T*/E} = \overline{\omega}_{T*} - \overline{\omega}_E \qquad (11\text{-}99)$$

THUS, BY EQUATIONS (11-98) AND (11-99):

$$\overline{A}_M = {}^{T*E}\dot{U}_M \; \overline{T}^*_1 + \overline{\omega}_{T*/E} \times {}^{E}\overline{V}_M + 2 \, \overline{\omega}_E \times {}^{E}\overline{V}_M + \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) \; . \quad (11\text{-}100)$$

CONSIDER THE SECOND TERM OF THE RIGHT SIDE OF EQUATION (11-100)

$$\overline{\omega}_{T*/E} = \overline{\omega}_{T*/G} + \overline{\omega}_{G/E} \; . \qquad (11\text{-}101)$$

SINCE THE $\overline{T}^*_1$ FRAME IS ORIENTED WITH RESPECT TO THE $\overline{G}_1$ FRAME BY A PITCH AND A YAW MATRIX OF EQUATION (11-89),

$$\overline{\omega}_{T*/G} = \dot{\theta}_{T*/G} \; \overline{G}_2 + \dot{\psi}_{T*/G} \; \overline{T}^*_3 \; . \qquad (11\text{-}102)$$

THE ANGULAR VELOCITY OF THE $\overline{G}_1$ FRAME WITH RESPECT TO THE $\overline{E}_1$ (EARTH) FRAME IS:

$$\overline{\omega}_{G/E} = \overline{\omega}_G - \overline{\omega}_E \; . \qquad (11\text{-}103)$$

THUS, BY USE OF EQUATION (11-101), THE SECOND TERM OF EQUATION (11-100) MAY BE EXPRESSED AS:

$$\overline{\omega}_{T*/E} \times {}^{E}\overline{V}_M = \overline{\omega}_{T*/G} \times {}^{E}\overline{V}_M + \overline{\omega}_{G/E} \times {}^{E}\overline{V}_M \; , \qquad (11\text{-}104)$$

WHERE, BY EQUATION (11-89)

$$\overline{G}_2 = S\psi_{T*/G}\ \overline{T}_1^* + C\psi_{T*/G}\ \overline{T}_2 \tag{11-105}$$

AND,

$$\omega_{T*/G} \times {}^E\overline{V}_M = \begin{bmatrix} \overline{T}_1^* & \overline{T}_2^* & \overline{T}_3^* \\ (\dot{\theta}_{T*/G}\ S\psi_{T/G} & C\ \dot{\theta}_{T/G})\ \theta_{T*/G}\ C\psi_{T*/G} & \dot{\psi}_{T*/G} \\ {}^EU_M & 0 & 0 \end{bmatrix}$$

$$= {}^EU_M\ \dot{\psi}_{T*/G}\ \overline{T}_2^* - \dot{\theta}_{T*/G}\ C\psi_{T*/G}\ {}^EU_M\ \overline{T}_3^* . \tag{11-107}$$

NEXT, CONSIDER THE TERM $\overline{\omega}_{G/E} \times E_V$. THE POSITION VECTOR OF THE MISSILE WITH RESPECT TO THE INERTIAL ORIGIN O EXPRESSED IN THE $\overline{G}_1$ FRAME IS, BY EQUATION (2-1), 2-2), AND (2-3):

$$\overline{R}_M = Z_{0'}^E\ \overline{E}_3 + (R_E + H_M)\ \overline{G}_3 \tag{11-108}$$

AND

$$\overline{V}_M = {}^E\dot{Z}_{0'}\ \overline{E}_3 + (\dot{R}_E + \dot{H}_M)\ \overline{G}_3 + \overline{\omega}_G \times \overline{R}_M = {}^E\overline{V}_M + \overline{\omega}_E \times \overline{R}_M = {}^E\overline{V}_M\ \omega_E \times \overline{R}_M \tag{11-109}$$

WHERE, BY EQUATION (9-38),

$$Z_{0'}^E = -\ 2AF\ \ SIN\ \lambda \tag{11-110}$$

AND,

$$R_E = A\left[1 + F\ \ SIN^2 \lambda\right] \tag{11-111}$$

BY EQUATIONS (11-103) AND (11-109)

$${}^E\dot{Z}_{0'}\ \overline{E}_3 + (\dot{R}_E + \dot{H}_M)\ \overline{G}_3 + \overline{\omega}_{G/E} \times \overline{R}_M = {}^E\overline{V}_M \tag{11-112}$$

WHERE,

$$\overline{\omega}_{G/E} = P_{G/E}^{T*}\ \overline{T}_1^* + Q_{G/E}^{T*}\ \overline{T}_E + R_{G/E}^{T*}\ \overline{T}_3^*$$

AND,

$$\bar{R}_M = Z_o^E (G_{13}^E \bar{G}_1 + G_{23}^E \bar{G}_2 + G_{33}^E \bar{G}_3) + (R_E + H_M)\bar{G}_3$$

OR,

$$\bar{R}_M = Z_o^E G_{13}^E \bar{G}_1 + Z_o^E G_{23}^E \bar{G}_2 + (Z_o^E G_{33}^E + R_E + H_M)\bar{G}_3 \quad . \quad (11\text{-}113)$$

BY EQUATIONS (11-113) AND (11-112)

$$\begin{pmatrix} \bar{G}_1 & \bar{G}_2 & \bar{G}_3 \\ P^G_{G/E} & Q^G_{G/E} & R^G_{G/E} \\ Z_o^E \ G_{13}^E & Z_o^E \ G_{23}^E & (Z_o^E \ G_{33}^E + R_E + H_M) \end{pmatrix}$$

$$= {}^E V_M \bar{T}_1^* - {}^E\dot{Z}_o \ G_{13}^E \ \bar{G}_1 + {}^E\dot{Z}_o \ G_{23}^E \ \bar{G}_2 + ({}^E\dot{Z}_o + \dot{H}_M + \dot{R}_E)\bar{G}_3 \quad (11\text{-}114)$$

BY EQUATIONS (11-110) AND (11-112)

$$^E\dot{Z}_o = (-2AF \cos\lambda) \dot{} \quad (11\text{-}115)$$

$$\dot{R}_E = (2AF \ G_{33}^E \cos\lambda) \dot{} \quad (11\text{-}116)$$

BY EQUATION (11-89)

$$T_1^* = C\psi_{T^*/G} \ C\theta_{T^*/G} \ \bar{G}_1 + S\psi_{T^*/G} \ G_2 - C\psi_{T^*/G} \ S\theta_{T^*/G} \ \bar{G}_3 \ . \quad (11\text{-}117)$$

EQUATING COMPONENTS OF EQUATION (11-114) BY USE OF (11-115), (11-116) AND (11-117),

$$Q^G_{G/E}\left[-2AF\,(G_{33}^E)^2 + A + AF\,(G_{33}^E)^2 + H_M\right] + R_{G/E}\,(2AF \ G_{33}^E \ G_{23}^E)$$

$$= {}^E V_M \ C\psi_{T^*/G} \ C\theta_{T^*/G} + 2AF \ G_{33}^E \cos\dot{\lambda} \ \dot{\lambda} G_{13}^E$$

$$- P^G_{G/E}\left[-AF\,(G^E_{33})^2 + A + H_M\right] + R^G_{G/E}\left[(-2AF\,G^E_{33}\,G^E_{13})\right]$$

$$= {}^EV_M\;S\psi_{T*/G} + 2AF\,G^E_{33}\;\cos\lambda\,\dot{\lambda}\;G^E_{23}$$

$$- P^G_{G/E}\;2AF\;G^E_{33}\;G^E_{23} + Q^G_{G/E}\;2AF\;G^E_{13}\;G^E_{33}$$

$$= -{}^EV_M\,C\psi_{T*/G}\;S\,\theta_{T*/G} + 2AF\cos\lambda\,\dot{\lambda} - 2AF\;G^E_{33}\cos\lambda\,\dot{\lambda} - \dot{H}_M\,. \tag{11-118}$$

SOLVING EQUATIONS (11-118) FOR $P^G_{G/E}$, $Q^G_{G/E}$, AND $R^G_{G/E}$ WHERE $R^G_{G/E}$ IS SPECIFIED BY CONDITION $R^G_G = 0$, THAT IS

$$\bar{\omega}_G = P^G_G\,\bar{G}_1 + Q^G_G\,\bar{G}_2 + 0\,\bar{G}_3 = \bar{\omega}_E + \bar{\omega}_{G/E} = \omega_E\,\bar{E}_3 + P^G_{G/E}\,\bar{G}_1 + Q^G_{G/E}\,\bar{G}_2 + R^G_{G/E}\bar{G}_3\;: \tag{11-119}$$

$$R^G_{G/E} = -\omega_E\;\bar{E}_3\cdot\bar{G}_3$$

$$= \omega_E\;\sin\lambda \tag{11-120}$$

$$P^G_{G/E} = \frac{\omega_E\,\sin\lambda\,(-2AF\,G^E_{33}\,G^E_{13}) + {}^EV_M\,\sin\psi_{T*/G} + 2AF\,G^E_{33}\,\cos\lambda\,\dot{\lambda}\,G^E_{23}}{AF\,(G^E_{33})^2 - A - H_M} \tag{11-121}$$

$$Q^G_{G/E} = \frac{\omega_E\,\sin\lambda(2AF\,G^E_{33}\,G^E_{23}) + {}^EV_M\,\cos\psi_{T*/G}\,\cos\theta_{T*/G} + 2AF\,G^E_{33}\,\cos\lambda\,\dot{\lambda}\,G^E_{13}}{-AF\,(G^E_{33})^2 + A + H_M)} \tag{11-122}$$

A COORDINATE TRANSFORMATION OF THE ABOVE THREE COMPONENTS FROM THE $\overline{G}_i$ FRAME TO THE $\overline{T}^*_i$ FRAME YIELDS:

$$P^{T*}_{G/E} = P^G_{G/E}\, C\,\psi_{T*/G}\, C\,\theta_{T*/G} + Q^G_{G/E}\, S\,\psi_{T*/G} - R^G_{G/E}\, C\psi_{T*/G}\, S\,\theta_{T*/G}$$

$$Q^{T*}_{G/E} = P^G_{G/E}\, S\psi_{T*/G}\, C\,\theta_{T*/G} + Q^G_{G/E}\, C\psi_{T*/G} + R^G_{G/E}\, S\psi_{T*/G}\, S\,\theta_{T*/G}$$

$$R^{T*}_{G/E} = P^G_{G/E}\, S\,\theta_{T*/G} + R^G_{G/E}\, C\,\theta_{T*/G} \qquad (11\text{-}123)$$

THE TERM $\overline{\omega}_{G/E} \times {}^E\overline{V}_M$ OF EQUATION (11-104) IS

$$\overline{\omega}_{G/E} \times {}^E V_M = \begin{vmatrix} T^*_1 & T^*_2 & T^*_3 \\ P^{T*}_{G/E} & Q^{T*}_{G/E} & R^{T*}_{G/E} \\ {}^E U^{T*}_M & 0 & 0 \end{vmatrix}$$

OR,

$$\omega_{G/E} \times {}^E\overline{V}_M = {}^E U^{T*}_M\; R^{T*}_{G/E}\; \overline{T}^*_2 - {}^E U^{T*}_M\; Q^{T*}_{G/E}\; \overline{T}^*_3 .$$

THE CORIOLIS ACCELERATION TERM $2\,\overline{\omega}_E \times {}^E\overline{V}_M$ OF EQUATION (11-100) IS DESIRED IN COMPONENT FORM IN THE $\overline{T}^*_i$ FRAME. THE ORIENTATION OF THE $\overline{T}^*_i$ FRAME WITH RESPECT TO THE $\overline{E}_i$ FRAME IS OBTAINED AS THE PRODUCT OF THE FOLLOWING TWO MATRICES:

$$H_1 = \begin{pmatrix} S\,\lambda & CL & 0 \\ -S\lambda\, CL & -S\lambda\, SL & C\,\lambda \\ C\lambda\, CL & C\lambda\, SL & S\,\lambda \end{pmatrix} E_i \qquad (11\text{-}125)$$

AND ORIENTING THE $\overline{T}^*_1$ FRAME WITH RESPECT TO THE $\overline{H}_1$ FRAME THROUGH A YAW, PITCH, ZERO ROLL SEQUENCE -

$$\overline{T}^*_1 = \begin{pmatrix} C\,\theta_{T^*/H}\; C\,\psi_{T^*/H} & C\,\theta_{T^*/H}\; S\,\psi_{T^*/H} & -S\,\theta_{T^*/H} \\ -S\,\psi_{T^*/H} & C\,\psi_{T^*/H} & 0 \\ S\,\theta_{T^*/H} & S\,\theta_{T^*/H}\; S\,\psi_{T^*/H} & C\,\theta_{T^*/H} \end{pmatrix}_{H_1} \tag{11-126}$$

THE PRODUCT MATRIX

$$\overline{T}^*_i = (\overline{T}^{*E}_{ij})\ \varepsilon_1$$

HAS THE NEEDED ELEMENTS T^{*E}_{i3} AS

$$\overline{T}^*_{13} = C\,\theta_{T^*/H}\ S\,\psi_{T^*/H}\ C\,\lambda - S\,\theta_{T^*/H}\ S\,\lambda$$

$$\overline{T}^{*E}_{23} = C\psi_{T^*/H}\ C\,\lambda$$

$$\overline{T}^{*E}_{33} = S\,\theta_{T^*/H}\ S\psi_{T^*/H}\ C\,\lambda + C\,\theta_{T^*/H}\ S\,\lambda, \tag{11-127}$$

CONSEQUENTLY

$$\varepsilon_3 = T^E_{13}\ \overline{T}^*_1 + T^E_{23}\ \overline{T}^*_2 + T^E_{33}\ \overline{T}^*_3$$

OR,

$$2\,\overline{\omega}_E \times {}^E\overline{V}_M = 2\,\overline{\omega}_E \begin{pmatrix} \overline{T}^*_1 & \overline{T}^*_2 & \overline{T}^*_3 \\ T^{*E}_{13} & T^{*E}_{23} & T^{*E}_{33} \\ -{}^{\varepsilon}U_M & 0 & 0 \end{pmatrix}$$

$$= 2\omega_E \; {}^E U_M^{T*} \left[\left(S\,\theta_{T*/H}\, S\psi_{T*/H}\, C\lambda + C\,\theta_{T*/H}\, S\lambda \right) \overline{T}_1^* + C\,\psi_{T*/H}\; C\lambda\, \overline{T}_3 \right] \tag{11-128}$$

IT CAN BE SHOWN BY EQUATING THE MATRICES

$$\overline{T}_1 = M_Y\,(\psi_{T*/G})\; M_P\,(\theta_{T*/G})\; M_Y\,(\psi_{G/H})\,\overline{H}_1$$

AND,

$$\overline{T}_1 = M_P\,(\theta_{T*/H})\; M_Y\,(\psi_{T*/H})\,\overline{H}_1$$

THAT

$$\theta_{T*/H} = \theta_{T*/G}\,. \tag{11-129}$$

FROM THE FOREGOING EQUATIONS ONE MAY WRITE THE MISSILE TRANSLATIONAL EQUATIONS OF EQUATION (11-100) AS:

$$\overline{A}_M = \frac{\overline{F}_A}{M} + \frac{\overline{F}_T}{M} + \frac{\overline{F}_{N_G}}{M}\,. \tag{11-130}$$

OR,

$$ {}^{T*E}\dot{U}_M \;\; \overline{T}_1^* + \overline{\omega}_{T*/E} \times {}^E\overline{V}_M + 2\overline{\omega}_E \times {}^E\overline{V}_M = \frac{\overline{F}_A}{M} + \frac{\overline{F}_T}{M} \tag{11-131}$$

$$+ \left[\frac{F_{N_G}}{M} - \overline{\omega}_E \times \left(\overline{\omega}_E \times \overline{R}_M \right) \right]\,.$$

IN COMPONENT FORM:

$$ {}^{T*E}\dot{U}_M\, \overline{T}_1^* + {}^E U_M^{T*} \left[\dot{\psi}_{T*/G} + R_{G/E}^{T*} + 2\,\omega_E\,(S\,\theta_{T*/H}\, S\psi_{T*/H}\; C\lambda + C\,\theta_{T*/H}\; S\lambda) \right] \overline{T}_2^*$$

$$+ {}^E U_M^{T*} \left(-\dot{\theta}_{T*/G}\, C\psi_{T*/G} - Q_{G/E}^{T*} + 2\,\omega_E\, C\,\psi_{T*/H}\; C\lambda \right) \;\; T_3^*$$

$$= \frac{\overline{F}_A}{M} + \frac{\overline{F}_T}{M} + \overline{F}_{GE}\,. \tag{11-132}$$

where the effective gravity term

$$\overline{F}_{GE} = \frac{\overline{F}_{N_G}}{M} - \overline{\omega}_E \times (\overline{\omega}_E \times \overline{R}_M) .$$

VARIOUS REFERENCE FRAMES USED IN ORIENTING AERODYNAMIC FORCES

Five reference frames in which the aerodynamic forces are often expressed are:

1. Body-fixed axes $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$

2. Wind axes oriented with respect to $\overline{B}_1$ through an angle of attack in pitch α, and an angle of attack in yaw β.

3. Stability axes

For missiles having symmetry about two planes

4. An axial (chord) force along the missile longitudinal axis and a normal force.

5. A drag force along relative wind vector and a lift force perpendicular to the drag force.

Let F_A represent the resultant aerodynamic force, it may be broken up into components in accordnace with the above reference frames as follows:

1. Three components along body-fixed axes.

$$\overline{F}_A = F^B_{A_1} \overline{B}_1 + F^B_{A_2} \overline{B}_2 + F^B_{A_3} \overline{B}_3 \qquad (12\text{-}1)$$

2. Three components along wind axes are defined in terms of an angle of attack in pitch , and angle of attack in yaw (also called sideslip angle), thus

$$\overline{F}_A = F^W_{A_1} \overline{W}_1 + F^W_{A_2} \overline{W}_2 + F^W_{A_3} \overline{W}_3 \qquad (12\text{-}2)$$

IF THE WIND FRAME $\overline{W}_1$ IS CONSIDERED ORIENTED WITH RESPECT TO THE BODY FRAME THROUGH A PITCH $\Theta_{W/B}$ AND A YAW $\psi_{W/B}$, ONE OBTAINS

$$\overline{W}_i = M_Y(\psi_{W/B}) \; M_P(\Theta_{W/B}) \, \overline{B}_i \, . \tag{12-3}$$

SINCE THERE IS LITTLE CONSISTENCY IN THE LITERATURE REGARDING THE SIGNS OF α AND β, THIS PROBLEM MAY BE HANDLED CONSISTENTLY FOR ANY INDIVIDUAL MISSILE STUDY. FOR EXAMPLE, LET $\Theta_{W/B} = -\alpha$ (12-4)

AND $\psi_{W/B} = -\beta$

THEN EQUATION (12-3) BECOMES

$$\overline{W}_i = M_Y(-\beta) \; M_P(-\alpha) \, \overline{B}_i \tag{12-5}$$

AS SHOWN IN FIGURE 12-1. UPON MULTIPLICATION, EQUATION (12-5) BECOMES:

$$\begin{pmatrix} \overline{W}_1 \\ \overline{W}_2 \\ \overline{W}_3 \end{pmatrix} = \begin{pmatrix} c\beta \, c\alpha & -s\beta & c\beta \, s\alpha \\ +s\beta \, c\alpha & c\beta & +s\alpha \, s\beta \\ -s\alpha & 0 & c\alpha \end{pmatrix} \begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \overline{B}_3 \end{pmatrix} . \tag{12-6}$$

IF HOWEVER, THE BODY FRAME IS CONSIDERED TO BE ORIENTED WITH RESPECT TO THE WIND FRAME THROUGH A YAW $\psi_{B/W} =$ AND A PITCH $\Theta_{B/W} = \alpha$, AS SHOWN IN FIGURE 12-2, ONE OBTAINS $\overline{B}_i = M_P(\alpha) \, M_Y(\beta) \; \overline{W}_i$ (12-7)

$$\begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \overline{B}_3 \end{pmatrix} = \begin{pmatrix} c\beta \, c\alpha & +s\beta \, c\alpha & -s\alpha \\ -s\beta & c\beta & 0 \\ c\beta \, s\alpha & +s\beta \, s\alpha & c\alpha \end{pmatrix} \begin{pmatrix} \overline{W}_1 \\ \overline{W}_2 \\ \overline{W}_3 \end{pmatrix} \tag{12-8}$$

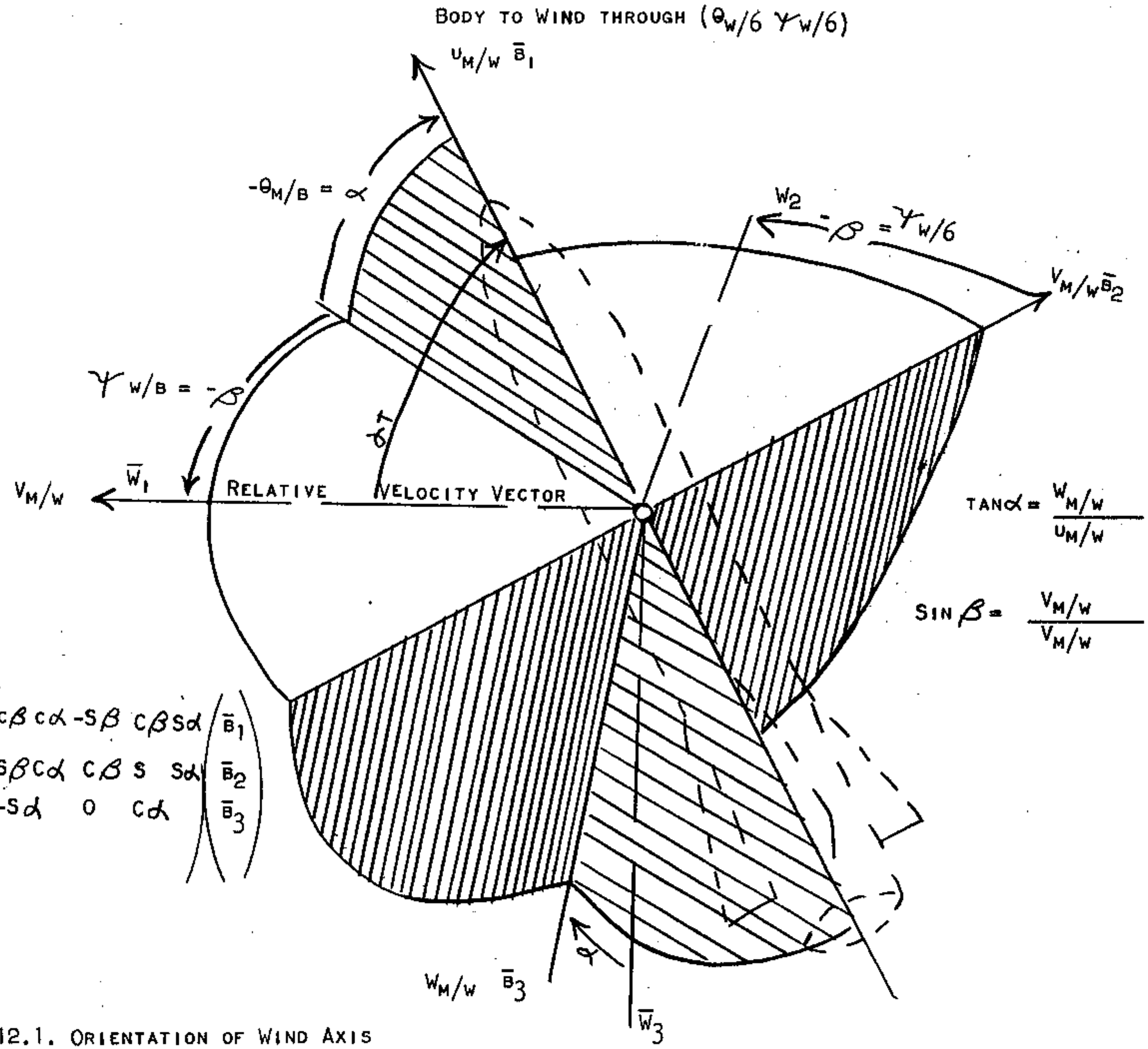
THE MATRIX OF EQUATION (12-8) IS THE TRANSPOSE (INTERCHANGE OF ROWS AND COLUMNS) OF THE MATRIX OF EQUATION (12-6).

THE ANGLES AS DEFINED IN FIGURE 12-2 WILL BE USED IN THIS PAPER HEREAFTER.

THE INERTIAL ANGULAR VELOCITIES OF THE WIND FRAME MAY BE OBTAINED FROM $\overline{\omega}_B = \overline{\omega}_W + \dot{\beta} \, \overline{W}_3 + \dot{\alpha} \, \overline{B}_2$, (12-9)

OR IN TERMS OF INERTIAL ORTHOGONAL COMPONENTS ABOUT WIND AND BODY AXES:

$$P_B \overline{B}_1 + Q_B \overline{B}_2 + R_B \overline{B}_3 = P_W \overline{W}_1 + Q_W \overline{W}_2 + R_W \overline{W}_3 + \dot{\beta} \, \overline{W}_3 + \dot{\alpha} \, \overline{B}_2 \, . \tag{12-10}$$



$$\tan\alpha = \frac{W_{M/W}}{U_{M/W}}$$

$$\sin\beta = \frac{V_{M/W}}{V_{M/W}}$$

$$\begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \end{pmatrix} = \begin{pmatrix} c\beta c\alpha & -S\beta & C\beta S\alpha \\ S\beta C\alpha & C\beta & S\ S\alpha \\ -S\alpha & 0 & C\alpha \end{pmatrix} \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix}$$

FIG. 12.1. ORIENTATION OF WIND AXIS

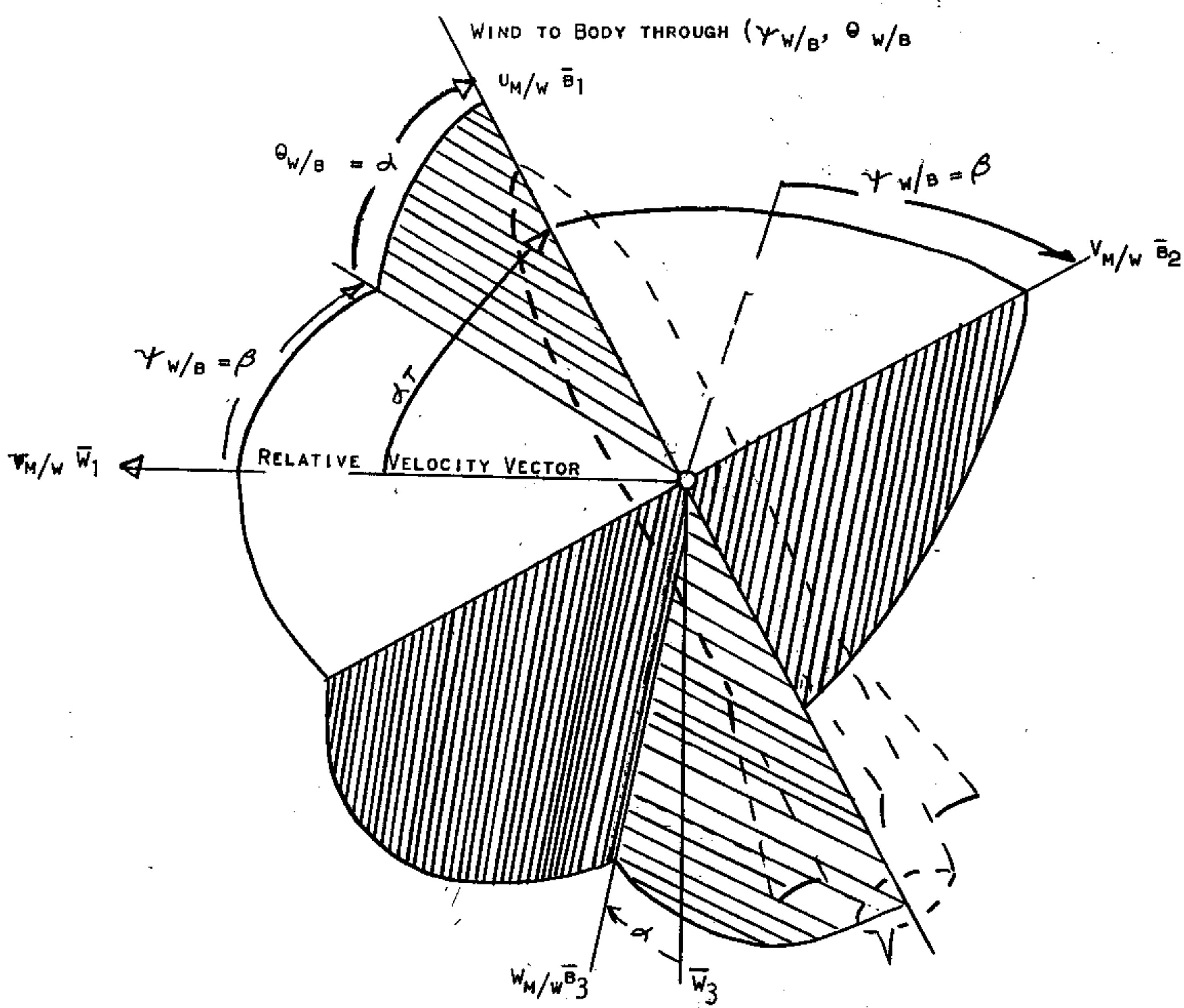


Fig. 12.2. Orientation of Body Axis with Respect to Wind Axis.

DOTTING EQUATION (12-10) BY $\overline{W}_I$ RESPECTIVELY AND UTILIZING EQUATION (12-8) ONE OBTAINS

$$P_W = P_B \, c\beta c\alpha - s\beta Q_B + s\alpha c\alpha R_B + s\beta\dot{\alpha}$$

$$Q_W = P_B \, c\alpha s\beta + Q_B \, c\beta + s\alpha s\beta R_B - c\beta\dot{\alpha}$$

$$R_W = P_B \, (- s\alpha + R_B \, c\alpha - \dot{\beta}$$

SINCE, GENERALLY, IN SIMULATION STUDIES P_W, Q_W, R_W ARE NOT KNOWN BUT P_B, Q_B AND R_B ARE KNOWN, ONE MAY OBTAIN THE DIRECTION COSINE MATRIX OF EQUATION (12-8) FROM THE RELATIVE VELOCITY VECTOR COMPONENTS. THE VELOCITY OF THE MISSILE WITH RESPECT TO THE WIND IS GIVEN AS

$$\overline{V}_{M/W} = V_{M/W} \; \overline{W}_1 = U_{M/W} \; \overline{B}_1 + V_{M/W} \, \overline{B}_2 + W_{M/W} \, \overline{B}_3, \quad (12\text{-}12)$$

WHEN $U_{M/W}$, $V_{M/W}$, AND $W_{M/W}$ ARE RELATIVE VELOCITY COMPONENTS ALONG BODY AXES.

DOTTING EQUATION (12-12) BY $\overline{B}_I$ RESPECTIVELY AND UTILIZING EQUATION (12-8).

$$V_{M/W} \; \overline{W}_1 \cdot \overline{B}_1 = V_{M/W} \; c\beta c\alpha = U_{M/W} \qquad (12\text{-}13)$$

$$V_{M/W} \; \overline{W}_1 \cdot \overline{B}_2 = V_{M/W} \; (-s\beta) = V_{M/W} \qquad (12\text{-}14)$$

$$V_{M/W} \; \overline{W}_1 \cdot \overline{B}_3 = V_{M/W} \; c\beta s\alpha = W_{M/W} \qquad (12\text{-}15)$$

BY EQUATIONS (12-13) AND (12-15)

$$\tan\alpha = \frac{W_{M/W}}{U_{M/W}} \quad (12\text{-}16) \qquad\qquad \sin\beta = \frac{-V_{M/W}}{U_{M/W}} \quad (12\text{-}17)$$

AND SMALL ANGLE APPROXIMATIONS ON α AND β YIELD THE FAMILIAR EXPRESSIONS

$$\alpha \simeq \frac{W_{M/W}}{U_{M/W}} \qquad (12\text{-}18)$$

$$\beta \simeq \frac{V_{M/W}}{U_{M/W}} \qquad (12\text{-}19)$$

SINCE $V_{M/W} \simeq u_{M/W}$ WHEN α AND β ARE SMALL.

THE DIRECTION COSINE BETWEEN THE MISSILE LONGITUDINAL AXIS $\overline{B}_1$ AND THE RELATIVE VELOCITY VECTOR $\overline{W}_1$ IS, BY FIGURE 12-2, $\overline{W}_1 \cdot \overline{B}_1 = \cos\alpha_T$

$$= \cos\alpha \cos\beta \tag{12-20}$$

EXPANDING

$$1 - \frac{\alpha^2_T}{2!} + \frac{\alpha^4_T}{4!} - \ldots \left(1 - \frac{\alpha^2}{2!} + \ldots\right)\left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \ldots\right)$$

$$1 - \frac{\alpha^2_T}{2!} = 1 - \frac{\alpha^2}{2!} - \frac{\beta^2}{2!}$$

$$\alpha^2_T \simeq \alpha^2 + \beta^2 \tag{12-21}$$

BY EQUATIONS (12-18) AND (12-19)

$$\alpha^2_T \simeq \frac{w^2_{M/W}}{U^2_{M/W}} + \frac{V^2_{M/W}}{U^2_{M/W}} \tag{12-22}$$

DOTTING EQUATION (12-12) BY ITSELF

$$V^2_{M/W} = u^2_{M/W} + v^2_{M/W} + w^2_{M/W} \tag{12-23}$$

$$V^2_{M/W} = u^2_{M/W} \left[1 + \frac{v^2_{M/W}}{u^2_{M/W}} + \frac{w^2_{M/W}}{u^2_{M/W}}\right]$$

BY EQUATION (12-22)

$$V^2_{M/W} = u^2_{M/W} \left[1 + \alpha^2_T\right] . \tag{12-24}$$

ONE OTHER EXPRESSION THAT MAY BE USEFUL IS OBTAINED BY DOTTING EQUATION (12-12) BY $\overline{B}_1$.

$$V_{M/W} \cos(B_1, W_1) = V_{M/W} \cos \quad _T = u_{M/W}. \tag{12-25}$$

(3) THE AERODYNAMIC FORCES ARE SOMETIMES GIVEN ALONG STABILITY AXES AS $\overline{F}_A = F^{\sigma}_{A1}\overline{\sigma}_1 + F^{\sigma}_{A2}\overline{\sigma}_2 + F^{\sigma}_{A2}\overline{\sigma}_3$, (12-26)

WHERE THE $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ UNIT VECTORS ARE DEFINED AS THE INTERMEDIATE SET OF AXES IN GOING FROM THE WIND FRAME $\bar{W}_i$ TO THE BODY FRAME $\bar{B}_i$, (E.G.) $\bar{\sigma}_i = M_Y(\beta)\, w_i$.

(4) FOR MISSILES HAVING TWO PLANES OF SYMMETRY, THE WIND TUNNEL DATA MAY BE GIVEN AS FORCE COMPONENTS LYING IN THE PLANE CONTAINING THE RELATIVE VELOCITY VECTOR AND THE LONGITUDINAL MISSILE AXIS. FOR A MISSILE FLYING AS SHOWN IN FIGURE 12-3.

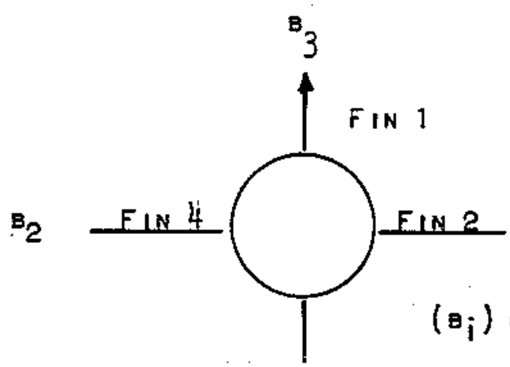


FIG. 12-3. $(B_i) = (F_i)$

2-PLANE SYMMETRICAL MISSILE FLYING INTO PAPER.

THE B_i FRAME EQUALS THE F_i FRAME.

FOR A CRUCEFORM MISSILE FLYING AS SHOWN IN FIGURE 12-4.

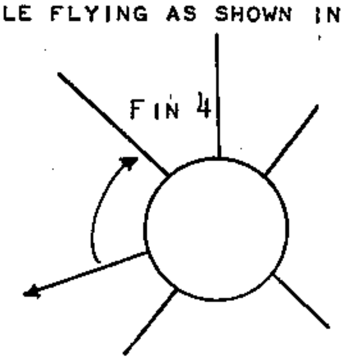


FIG. 12-4. 2-PLANE SYMMETRICAL MISSILE FLYING INTO PAPER.

THE FIN FRAME IS GIVEN AS

$$\bar{F}_i = M_R(45°)\, \bar{B}_i \,. \tag{12-27}$$

IF A REFERENCE FRAME M_i NOW DEFINED AS THE RESULTING FRAME AFTER A ROLL ABOUT THE $F_1 = B_1$, AXIS AS SHOWN IN FIGURE 12-5, ONE OBTAINS THE M_i FRAME WITH RESPECT TO THE F_i FRAME AS

$$\bar{M}_i = M_R(\phi_A)\, \bar{F}_i \tag{12-28}$$

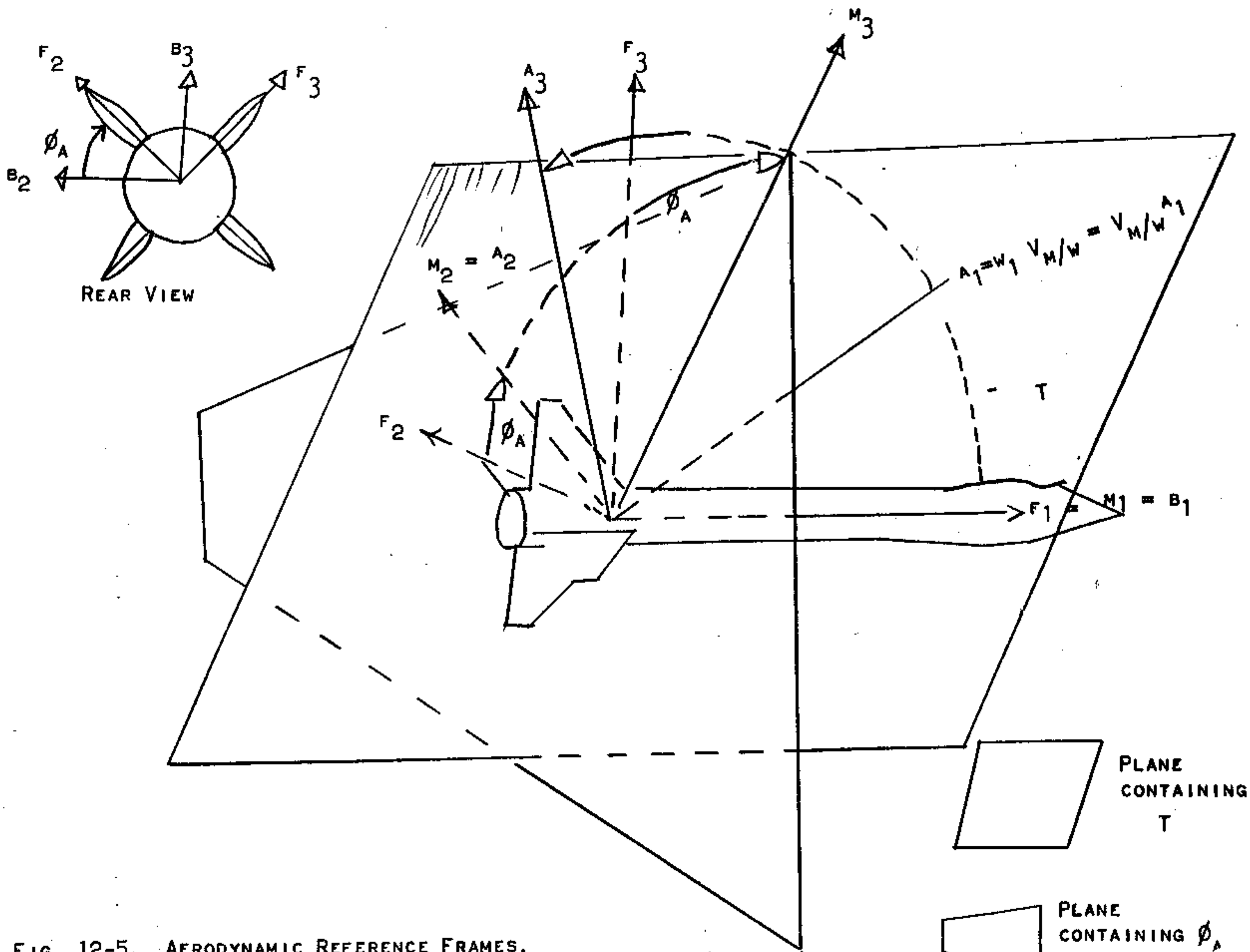


FIG. 12-5. AERODYNAMIC REFERENCE FRAMES.

$$\begin{pmatrix} \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C\,\phi_A & +S\,\phi_A \\ 0 & -S\,\phi_A & C\,\phi_A \end{pmatrix} \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{pmatrix}. \qquad (12\text{-}29)$$

BY FIGURE 12-5.

$$\bar{M}_2 = \frac{\bar{W}_1 \times \bar{M}_1}{\left|\bar{W}_1 \times \bar{M}_1\right|} = \frac{\bar{M}_3 \times \bar{W}_1}{\left|\bar{M}_3 \times \bar{W}_1\right|} \qquad (12\text{-}30)$$

WHERE $\bar{M}_2$ IS THE UNIT VECTOR NORMAL TO THE PLANE CONTAINING THE LONGITUDINAL BODY AXIS AND THE RELATIVE WIND VECTOR. ASSUMING THE SIDE FORCE (THE FORCE IN THE $\bar{M}_2$ DIRECTION) IS NEGLIGIBLE ONE MAY WRITE THE AERODYNAMIC FORCE AS TWO COMPONENTS.

$$\bar{F}_A = F^M_{AL}\ \bar{M}_1 + F^M_{A3}\ \bar{M}_3 \qquad (12\text{-}31)$$

WHERE F^M_{AL} IS OFTEN CALLED AN <u>AXIAL FORCE</u> AND F^M_{A3} THE <u>NORMAL FORCE</u>.

THE ANGLE BETWEEN THE $\bar{M}_1$ FRAME AND THE BODY-FIXED FIN FRAME, $\phi_{M/F}$, REFERRED TO IN THIS PAPER AS THE AERODYNAMIC ROLL ANGLE ϕ_A IS GIVEN AS

$$\bar{M}_3 . \bar{F}_3 = \text{Cos } \phi_A. \qquad (12\text{-}32)$$

METHODS OF OBTAINING ϕ_A ARE CONSIDERED. BY DEFINITION OF $\bar{M}_1$

$$\bar{M}_1 \times \bar{M}_2 = \bar{M}_3 \qquad (12\text{-}33)$$

BY EQUATIONS(12-30),(12-32) AND(12-33)

$$\left\{ \frac{\bar{M}_1 \times (\bar{W}_1 \times \bar{M}_1)}{\left|\bar{W}_1 \times \bar{M}_1\right|} \right\} . \bar{F}_3 = \text{Cos } \phi_A \qquad (12\text{-}34)$$

OR SINCE $\bar{W}_1 \times \bar{M}_1 = + \text{SIN}\alpha_T$

$$\left\{ \frac{(\bar{M}_1 . \bar{M}_1)\ \bar{W}_1 - (\bar{M}_1 .\ W_1)\ \bar{M}_1}{+ \text{SIN } \alpha_T} \right\} . \bar{F}_3 = \text{Cos } \phi_A$$

$$\frac{(\bar{M}_1 . \bar{M}_1)\ \bar{W}_1 . \bar{F}_3 - (\bar{M}_1 . \bar{W}_1)\ (\bar{M}_1 . \bar{F}_3)}{+ \text{SIN}\alpha_T} = \text{Cos } \phi_A \qquad (12\text{-}35)$$

AND BY EQUATION (12-29) $\overline{M}_1 . F_3 = 0$

EQUATION (12-35) BECOMES

$$\frac{\overline{W}_1 . \overline{F}_3 = \cos \phi_A}{+ \sin\alpha_T} \tag{12-36}$$

IF THE DIRECTION COSINE $\overline{W}_1.\overline{F}_3$ IS OBTAINED AS A FUNCTION OF α, β THEN BY EQUATION (12-27) AND EQUATION (12-7).

$$\overline{F}_1 = M_R \ (45°) \ (M_p \ (\alpha) \ M_Y \ (\beta) \ \overline{W}_1 \tag{12-37}$$

WHICH UPON MULTIPLICATION GIVES

$$\begin{pmatrix} \overline{F}_1 \\ \overline{F}_2 \\ \overline{F}_3 \end{pmatrix} = \begin{pmatrix} C\alpha C\beta & C\alpha S\beta & -S\alpha \\ S45° \ S\alpha C\beta - C\alpha S\beta & S45° \ S\alpha S\beta + C45° \ C\beta & S45° \ C\alpha \\ C45° \ S\alpha C\beta - S45° S\beta & C45° \alpha S\beta - S45° \ C\beta & C45° \ C\alpha \end{pmatrix} \begin{pmatrix} \overline{W}_1 \\ \overline{W}_2 \\ \overline{W}_3 \end{pmatrix}$$

BY EQUATIONS (12-36) AND (12-37)

$$\cos \phi_A = \frac{C \ 45° \ S \ \ C\beta + S45 S\beta .}{+ S \alpha_T} \tag{12-38}$$

IF THE RELATIVE VELOCITY VECTOR IS CONSIDERED

$$\overline{V}_{M/W} = V_{M/W} \overline{W}_1 = u_{M/W} \ \overline{F}_1 + v_{M/W} \overline{F}_2 + w_{M/W} \overline{F}_3 \tag{12-39}$$

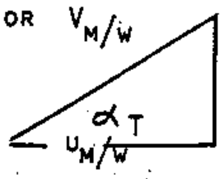
AND DOTTING BY $\overline{F}_3$: $\quad V_{M/W} \ \overline{W}_1 . \ \overline{F}_3 = w_{M/W}$

$$\text{OR } \overline{W}_1 . \overline{F}_3 = \frac{w_{M/W}}{V_{M/W}} \tag{12-40}$$

$$\therefore \cos \phi_A = - \frac{W_{M/W}}{V_{M/W} \sin\alpha_T} \tag{12-41}$$

$$\text{BY EQUATION (12-39) } V_{M/W} \cos\alpha_T = u_{M/W} \tag{12-42}$$

OR



$$\sqrt{V^2_{M/W} - U^2_{M/W}}$$

AND

$$\text{Sin}\,\alpha_T = \pm \frac{\sqrt{V_{M/W} - U_{M/W}}}{V_{M/W}} \qquad (12\text{-}43$$

SUBSTITUTING EQUATIONS (12-43) INTO (12-41)

$$\text{Cos}\,\phi_A = \frac{-W_{M/W} \quad V_{M/W}}{V_{M/W} \left(V^2_{M/W} - U^2_{M/W}\right)^{1/2}} \qquad (12\text{-}44)$$

$$\text{Cos}\,\phi_A = \frac{W_{M/W}}{\left(V^2_{M/W} - U^2_{M/W}\right)^{1/2}}$$

EQUATIONS (12-38) AND (12-41) HAVE THE UNDESIRABLE QUALITY OF HAVING ZERO DIVISORS WHEN $\alpha_T = 0$ AND CONSEQUENT COMPUTER DIFFICULTIES. THUS, FOR STUDIES IN WHICH α_T IS EXPECTED TO TAKE ON THE VALUE OF ZERO, ANOTHER EXPRESSION IS DESIRABLE.

(5) THE FINAL SET OF AERODYNAMIC FORCES ON MISSILES HAVING TWO PLANES OF SYMMETRY ARE THOSE COMMONLY REFERRED TO AS A DRAG FORCE ALONG THE RELATIVE WIND VECTOR AND A LIFT FORCE PERPENDICULAR TO THE DRAG FORCE.

THE UNIT VECTORS DEFINING THIS AERODYNAMIC REFERENCE FRAME ARE DESIGNATED AS $\bar{A}_1$ AS SHOWN IN FIGURE 12-5. THE $\bar{A}_1$ VECTORS ARE ORIENTED WITH RESPECT TO THE $\bar{M}_1$ FRAME THROUGH A PITCH MATRIX.

$$\bar{A}_1 = M_P\,(\alpha_T)\,\bar{M}_1 \qquad (12\text{-}45)$$

AND BY EQUATION (12-28)

$$\bar{A}_1 = M_P\,(\alpha_T)\,M_R\,(\phi_A)\,\bar{F}_1 \qquad (12\text{-}46)$$

THE FORCE EQUATION IS

$$\bar{F}_A = F^A_{A_1}\ \bar{A}_1 + F^A_{A_3}\ \bar{A}_3$$

THUS, WHEN THE AERODYNAMIC FORCES ARE GIVEN IN THE AERODYNAMIC FRAME $\bar{A}_I$ AND THE TRANSLATIONAL ACCELERATION EQUATIONS ARE BEING SOLVED IN BODY AXES, THE DIRECTION COSINES OF EQUATION (12-46) CAN BE GENERATED FROM α_T AND ϕ_A OBTAINED FROM EQUATIONS (12-25) AND (12-41) RESPECTIVELY. AS POINTED OUT PREVIOUSLY, THESE EQUATIONS HAVE A ZERO DIVISOR WHEN TOTAL ANGLE OF ATTACK $\alpha_T = 0$.

GRAVITATIONAL FORCE (TRANSFORMATIONS)

IN REFERENCE 9 IT IS SHOWN THAT THE NEWTONIAN MASS ATTRACTION GRAVITATIONAL FORCE IS:

$$\bar{F}_{N_G} = F^K_2\ \bar{K}_2 + F^K_3\ \bar{K}_3 \qquad (13\text{-}1)$$

WHERE F^K_1 AND F^K_2 ARE FUNCTIONS OF ALTITUDE H_M AND GEOCENTRIC LATITUDE λ*. SINCE $\bar{K}_2$ AND $\bar{K}_3$ LIE IN THE SAME MERIDIAN PLANE AS $\bar{H}_2$ AND $\bar{H}_3$ (FIG. 13-1) IT IS

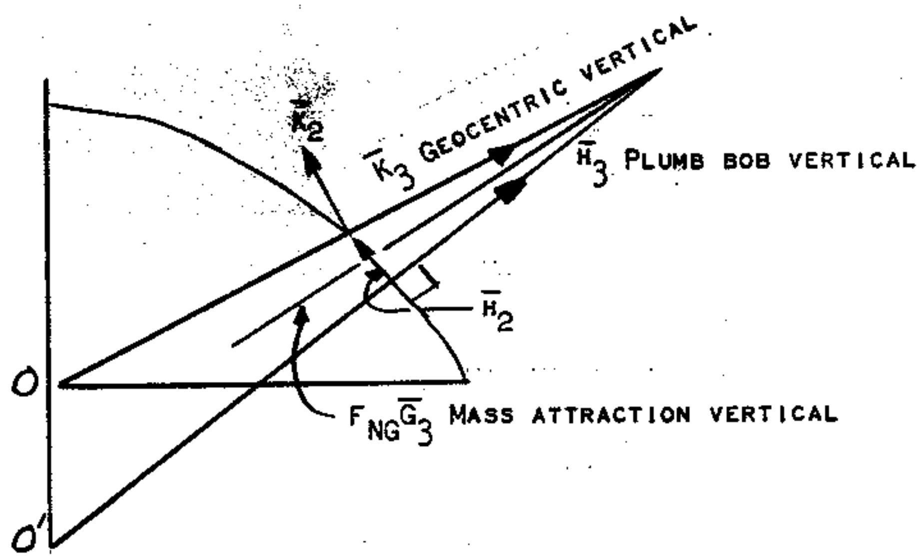


FIG. 13.1.

EVIDENT THAT $\bar{F}_{NG}$ CAN BE EXPRESSED AS:

$$\bar{F}_{NG} = F_2^H \bar{H}_2 + F_3^H \bar{H}_3 = F_{NG} \bar{G}_1 \tag{13-2}$$

DOTTING THE ABOVE EQUATION BY H_2

$$F_2^H = \bar{F}_{NG} \cdot \bar{H}_2 = F_2^K \bar{K}_2 \cdot \bar{H}_3 + \bar{F}_3^K \bar{K}_3 \cdot \bar{H}_3 \tag{13-3}$$

AND

$$F_3^H = \bar{F}_{NG} \cdot \bar{H}_3 = F_2^K \bar{K}_2 \cdot \bar{H}_3 + F_3^K \bar{K}_3 \cdot \bar{H}_3 \tag{13-4}$$

UTILIZING THE DIRECTION COSINE MATRIX ORIENTING THE $\bar{K}_1$ FRAME WITH RESPECT TO THE $\bar{H}_1$ FRAME OF EQUATION (9-10), ONE OBTAINS

$$\begin{aligned} \bar{K}_2 \cdot \bar{H}_2 &= C\,(\lambda - \lambda^*) \\ \bar{K}_3 \cdot \bar{H}_2 &= -S\,(\lambda - \lambda^*) \\ \bar{K}_2 \cdot \bar{H}_3 &= S\,(\lambda - \lambda^*) \\ K_3 \cdot H_3 &= C\,(\lambda - \lambda^*) \end{aligned} \tag{13-5}$$

BY EQUATION (13-4) AND EQUATION (13-5)

$$\begin{aligned} F_2^H &= F_2\, C\,(\lambda - \lambda^*) - F_3\, S\,(\lambda - \lambda^*) \\ F_3^H &= F_2\, S\,(\lambda - \lambda^*) + F_3\, C\,(\lambda - \lambda^*). \end{aligned} \tag{13-6}$$

DOTTING EQUATION (2) WITH $\bar{H}_2$ AND $\bar{H}_3$

$$\bar{G}_1 \cdot \bar{H}_2 = \frac{F_2^H}{F_{NG}} \tag{13-7}$$

$$\bar{G}_1 \cdot \bar{H}_3 = \frac{F_3^H}{F_{NG}} \tag{13-8}$$

$$\bar{G}_1 \cdot \bar{H}_1 = 0$$

WHERE $F_{NG} = \left[F_1^2 + F_2^2\right]^{1/2}$ (13-9)

THE THRUST FORCE MAY BE CONTROLLABLE IN MAGNITUDE AS WELL AS IN DIRECTION. THE CONTROL OF THE MAGNITUDE IS CURRENTLY BEST DONE BY MEANS OF VALVES AND LIQUID PROPELLANT. THE LATTER CONTROL MAY BE ACHIEVED THROUGH A GIMBALLED MOTOR OR OTHER SCHEMES. FOR A GIMBALLED MOTOR THE ANGULAR RELATIONSHIPS DEVELOPED IN SECTION 8 ARE APPLICABLE.

FOR A CONSTANT THRUST MALALIGNMENT NOT ACTING THROUGH THE MISSILE CENTER OF MASS, THE ORIENTATION OF THE $\bar{T}_1$ VECTOR (ALONG THE DIRECTION OF THE THRUST FORCE) WITH RESPECT TO THE BODY AXES $\bar{B}_1$ FOR AN EULER SEQUENCE YAW, PITCH IS SHOWN IN FIGURE 14.1.

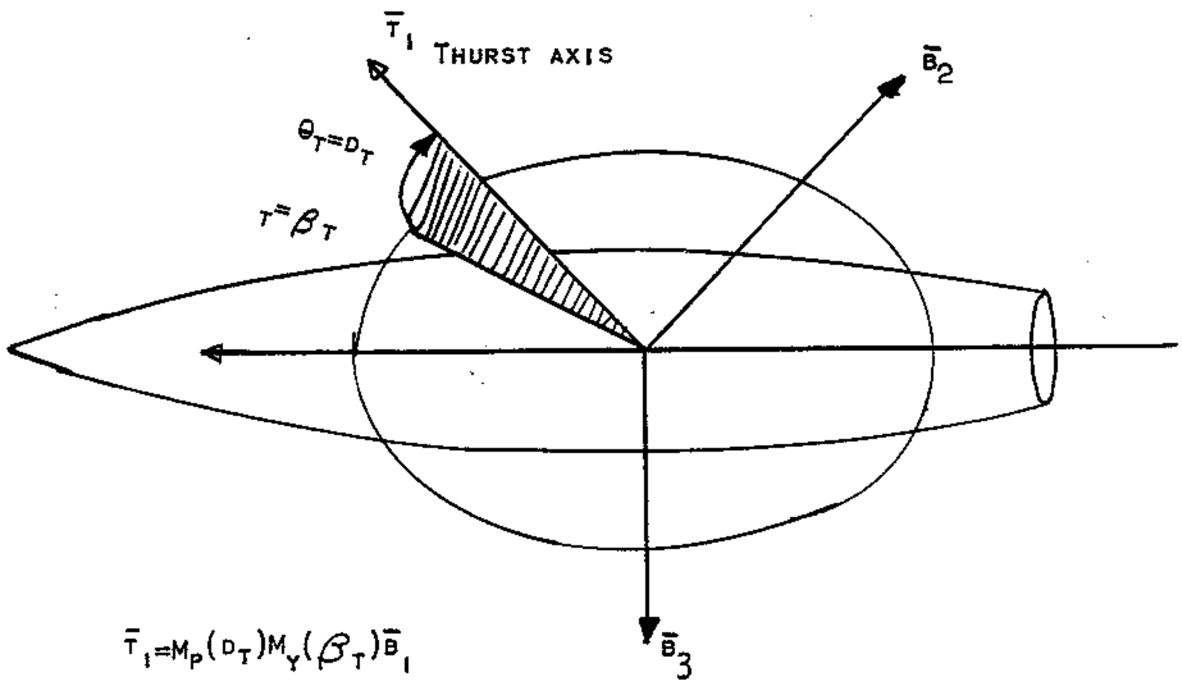


FIG. 14.1. THRUST MALALIGNMENT ANGLES.

THE DIRECTION COSINE MATRIX IS GIVEN BY FIGURE 3.2 AS:

$$T_{\bar{T}} = M_R\,(0)\; M_P\,(\theta_{T/B})\; M_Y\,(\psi_{T/B})\; \bar{B}_1 \; . \qquad (14\text{-}1)$$

THE COMPONENTS OF THE THRUST FORCE ALONG BODY AXIS ARE OBTAINED FROM

$$\bar{F}_T = F_T\,\bar{T}_1 = F^B_{1_T}\,\bar{B}_1 + F^B_{2_T}\,\bar{B}_2 + F^B_{3_T}\,\bar{B}_3 \; . \qquad (14\text{-}2)$$

DOTTING EQUATION (14-2) BY $\bar{B}_i$ RESPECTIVELY ONE OBTAINS

$$F^B_{1_T} = F_T \; C\,\theta_{T/B} \; C\,\psi_{T/B}$$

$$F^B_{2_T} = F_T \; C\,\theta_{T/B} \; S\,\psi_{T/B} \qquad (14\text{-}3)$$

$$F^B_{3_T} = S\,\theta_{T/B}$$

THE ROTATIONAL MOTION CAUSED BY THRUST MALALIGNMENT IS GIVEN BY:

$$\bar{R}_{T/M} \times \bar{F}_T = \begin{array}{ccc} \bar{B}_1 & \bar{B}_2 & \bar{B}_3 \\ X^B_{T/M} & Y^B_{T/M} & Z^B_{T/M} \\ F^B_{T_1} & F^B_{T_2} & F^B_{T_3} \end{array} \qquad (14\text{-}4)$$

WHERE $\bar{R}_{T/M}$ IS THE MOMENT ARM POSITION VECTOR FROM MISSILE C.G. TO POINT OF APPLICATION OF THRUST:

$$L_T = (Y^B_{T/M} \quad F^B_{T_3} - Z^B_{T/M} \quad F^B_{T_2}$$

$$M_T = (Z^B_{T/M} \quad F^B_{T_2} - Y^B_{T/M} \quad F^B_{T_1} \qquad (14\text{-}5)$$

DISCUSSION OF NOMENCLATURE

THE NOMENCLATURE UTILIZED IN THIS REPORT IS THE RESULT OF A NUMBER OF CHANGES OVER A FEW YEARS. AN ATTEMPT AT SUGGESTIVE, LOGICAL AND CONSISTENT SYMBOLS WHICH ARE AT THE SAME TIME SIMILAR (OR THE SAME) TO THOSE CURRENTLY ACCEPTED BY N. A. C. A. HAS BEEN SOUGHT. WHEN DEALING WITH AS BROAD AN ASPECT OF OVER-ALL WEAPON SYSTEMS AS THIS SERIES OF REPORTS WILL ATTEMPT TO DO, IT IS ONLY NATURAL THAT THE NOMENCLATURE PROBLEM INCREASES IN SIZE. WHEN DEALING WITH ANY ONE PARTICULAR MISSILE SYSTEM ONE NEED NOT EMPLOY SO ELABORATE A SCHEME OF SYMBOLS.

REFERENCE FRAME AS USED IN THIS REPORT REFERS TO A TRIPLE OF BASE VECTORS, WHEREAS COORDINATE SYSTEM REFERS TO A TRIPLE OF BASE VECTORS AND AN ORIGIN.

THE DIRECT DEFINITION OF COORDINATE SYSTEMS, AS SUCH, IS AVOIDED IN ORDER TO ADD FLEXIBILITY TO THE ORIGINS TO BE USED FOR THE VARIOUS REFERENCE FRAMES. CONSEQUENTLY, THE SPATIAL RELATIONSHIPS ARE CONSIDERED FROM THE STANDPOINT OF ANGULAR RELATIONS AND TRANSLATIONAL RELATIONS. THE ANGULAR ORIENTATIONS OF THE VARIOUS REFERENCE FRAMES OF INTEREST THROUGHOUT THE COMPLETE SYSTEM ARE DEVELOPED AND THE DIRECTION COSINE MATRICES AND THE EQUATIONS GENERATING THEM ARE REPEATEDLY DRAWN UPON FOR THE MANY PARTICULAR CASES ONE MAY CONSIDER.

THE FOUR SIMPLE BASIC METHODS REPEATEDLY ENCOUNTERED IN THE MISSILE LITERATURE FOR GENERATING THE DIRECTION COSINES FOR TRANSFORMATIONS OR ANGULAR MEASUREMENTS ARE DEVELOPED UNDER THE SECTION ON "GENERAL ANGULAR RELATIONSHIPS." HAVING ESTABLISHED THE RELATIONSHIPS (THE DIRECTION COSINE MATRICES) BETWEEN THE VARIOUS ORTHONORMAL BASE VECTORS ONE MAY PROCEED TO RELATE THE MANY SYSTEM VARIABLES MEASURED WITH RESPECT TO A PARTICULAR REFERENCE FRAME TO ANY OTHER REFERENCE FRAME.

POSITION VECTORS AND COORDINATES

WHEN COORDINATES ARE UTILIZED AT THE POSITION VECTOR LEVEL ONE MUST SPECIFY THE COORDINATES OF ONE POINT WITH RESPECT TO A SECOND POINT (TAKEN AS THE ORIGIN OF THE POSITION VECTOR) AND THE BASE VECTORS (DIRECTIONS) IN WHICH THE THREE VARIABLES ARE MEASURED. FOR EXAMPLE, THE POSITION VECTOR OF THE MISSILE CENTER OF MASS (POINT M) WITH RESPECT TO THE TARGET (POINT T) IS WRITTEN AS $\overline{R}_{M/T}$. IF THE SECOND SUBSCRIPT IS TAKEN AS THE ORIGIN OF THE SYSTEM SUCH AS AN INERTIAL POINT O AT THE CENTER OF A ROTATING, NON-TRANSLATING EARTH, THE SUBSCRIPT O IS LEFT OFF, I.E.,

$$\overline{R}_{M/O} = \overline{R}_M \ .$$

THE COMPONENTS OF A POSITION VECTOR ARE DESIGNATED AS X_i ($i = 1, 2, 3$) OR AS (X, Y, Z). A SUPERSCRIPT IN THE UPPER RIGHT HAND CORNER DESIGNATES THE BASIS VECTOR TO WHICH THE VECTOR IS REFERRED.

THUS,

$$\bar{R}_{M/T} = X^R_{M/T}\ \bar{R}_1 + Y^R_{M/T}\ \bar{R}_2 + Z^R_{M/T}\ \bar{R}_3 = X^R_{i\,M/T}\ \bar{R}_i\ ,$$

AND THE SYMBOL $Y^R_{M/T}$ DESIGNATES THE SECOND COORDINATE OF THE MISSILE AS MEASURED FROM THE TARGET AS ORIGIN FOR THE $\bar{R}_i$ BASE VECTORS, I.E. THE DISTANCE FROM POINT M TO POINT T IN THE R_2 UNIT VECTOR DIRECTION.

VELOCITY VECTORS AND COMPONENTS

THE COMPONENTS OF A VELOCITY VECTOR ARE DESIGNATED EITHER AS (U, V, W) OR AS (U_1, U_2, U_3). THE LETTER SUBSCRIPTS ON THE RIGHT HAND SIDE DESIGNATE THE VELOCITY OF ONE POINT WITH RESPECT TO A SECOND POINT E.G., $U_{M/T}$ IS THE FIRST COMPONENT OF THE VELOCITY OF THE MISSILE WITH RESPECT TO THE TARGET. A LEFT HAND SUPERSCRIPT DESIGNATES THE REFERENCE FRAME FROM WHICH THE TIME RATE OF CHANGE IS OBSERVED, AND A RIGHT SUPERSCRIPT DESIGNATES THE BASIC VECTORS INTO WHICH THE VECTOR IS RESOLVED. FOR EXAMPLE:

$$\dot{\bar{R}}_M = \bar{V}_M = U^B_{i\,M}\ \bar{B}_i = \frac{{}^E\partial\, \bar{R}_M}{\partial T} + \bar{\omega}_E \times \bar{R}_M$$

$$\bar{V}_M = {}^E\dot{X}_{i_M}\ \bar{E}_i + \bar{\omega}_E \times \bar{R}_M = {}^E U_{i_M}\ \bar{E}_i + \bar{\omega}_E \times \bar{R}_M$$

$$= {}^E\bar{V}_M + \bar{\omega}_E \times \bar{R}_M\ .$$

THE ABOVE EXPRESSIONS ARE EQUIVALENT, ALSO THE APPARENT VELOCITIES ARE EQUAL, THUS,

$${}^E\bar{V}_M = \frac{\partial\, {}^E\bar{R}_M}{\partial T} = {}^E X_{i_M}\ \bar{E}_i = {}^E U_{i_M}\ \bar{E}_i\ .$$

THE ABSENCE OF A LEFT HAND SUPERSCRIPT IMPLIES THE DERIVATIVES AS OBSERVED FROM AN INERTIAL FRAME.

IF A COORDINATE TRANSFORMATION IS MADE SAY FROM THE $\bar{E}_i$ BASIS TO THE $\bar{B}_i$ BASIS

$${}^E U_{i_M}\ \bar{E}_i = {}^E U^B_{i_M}\ \bar{B}_i\ ,$$

A RIGHT HAND SUPERSCRIPT DESIGNATES THE BASIS VECTORS TO WHICH THE VECTOR IS REFERRED. THE ABSENCE OF A RIGHT HAND SUPERSCRIPT IMPLIES THE SAME REFERENCE FRAME AS THE LEFT HAND SUPERSCRIPT. A SINGLE LETTER SUBSCRIPT IMPLIES M/O, THE INERTIAL ORIGIN AT THE CENTER OF THE EARTH. THE ADDITIONAL SYMBOL

IS USED FOR $\bar{\omega}_E \times \bar{R}_M$ OR SIMILAR TERMS,

$$\bar{\omega}_E \times \bar{R}_M = {}_{E/i}\ \bar{V}_M$$

WHICH DESIGNATES THE VELOCITY POINT M WOULD HAVE IF IT WERE FIXED IN THE ROTATING $\bar{E}_i$ FRAME AND OBSERVED BY AN INERTIAL OBSERVER. UTILIZING THE ABOVE EXPRESSIONS, THE COMPONENTS ARE WRITTEN AS:

$$\bar{V}_M = {}^{E}\bar{V}_M + {}_{E/i}\ \bar{V}_M$$

$$\bar{V}_M = {}^{E}U_{i_M}\ \bar{E}_i + {}_{E/i}\ U^{E}_{i_M}\ \bar{E}_i\ .$$

THE COMMONLY USED OPERATOR ON A VECTOR

$$\frac{D}{DT} = {}^{B}\frac{\partial}{\partial T} + \bar{\omega}_B \times$$

IS UTILIZED OCCASIONALLY AND SHOULD NOT BE CONFUSED WITH THE PARTIAL DERIVATIVE OF A SCALAR FUNCTION.

IN SUMMARY OF MEANING OF LEFT SUPERSCRIPT:

$${}^{B}\dot{\bar{R}} = ({}^{B}\dot{X}) = {}^{B}\bar{V} = ({}^{B}\dot{U}) ,$$

IF $B = i$ (INERTIAL OBSERVER) NO LEFT SUPERSCRIPT.

ACCELERATION VECTORS AND COMPONENTS

A SIMILAR MEANING IS GIVEN TO THE SECOND TIME DERIVATIVE AND THE SUPERSCRIPT FARTHEST TO THE LEFT DESIGNATES THE REFERENCE FRAME FROM WHICH THE LAST TIME DERIVATIVE IS OBSERVED. FOR EXAMPLE:

$${}^{CB}\ddot{\bar{R}} = ({}^{CB}\ddot{X}) = {}^{C}\dot{\overline{{}_{B}V}} = ({}^{CB}\dot{U}) = {}^{CB}\bar{A} = ({}^{CB}A_i) ,$$

DESIGNATES THE TIME RATE OF CHANGE AS OBSERVED BY AN OBSERVER ON THE $\bar{C}_i$ FRAME OF THE VECTOR ${}^{B}\dot{\bar{R}}$. THE VECTOR ${}^{B}\dot{\bar{R}}$ IS THE TIME RATE OF CHANGE OF THE POSITION VECTOR OBSERVED BY AN OBSERVER ON THE $\bar{B}_i$ FRAME.

ANGULAR VELOCITY VECTORS

THE SUBSCRIPTS ON THE RIGHT-HAND SIDE OF THE ANGULAR VELOCITY VECTOR FOR EXAMPLE $\bar{\omega}_{A/B}$ DESIGNATE THE ANGULAR VELOCITY OF REFERENCE FRAME A_i WITH RESPECT TO REFERENCE FRAME $\bar{B}_i$. IF $\bar{B}_i = \bar{\lambda}_i$ (AN INERTIAL FRAME) THEN

$$\bar{\omega}_{A/i} = \bar{\omega}_A$$

IS USED TO IMPLY WITH RESPECT TO AN INERTIAL FRAME. THE TIME DERIVATIVE OF AN ANGULAR VELOCITY VECTOR IS

$$ {}^{E}\dot{\bar{\omega}}_{A/B} = {}^{D}\dot{\bar{\omega}}_{A/B} + \bar{\omega}_{D/E} \times \bar{\omega}_{A/B} . $$

IF THE E_i FRAME IS AN INERTIAL FRAME, THAT IS, $E = \hat{1}$, THEN

$$ {}^{E}\dot{\bar{\omega}}_{A/B} = \dot{\bar{\omega}}_{A/B} = (\dot{P}_{A/B}) . $$

THE COMPONENTS OF THE ANGULAR VELOCITY VECTOR ARE DESIGNATED AS (P, Q, R) OR AS (P_1, P_2, P_3). SUPERSCRIPTS ARE USED AS DISCUSSED UNDER VELOCITY VECTORS.

DIRECTION COSINES AND DIRECTION COSINE MATRICES

DIRECTION COSINE MATRICES ARE DESIGNATED, AS AN EXAMPLE, AS $M_{B/G}$ THE DIRECTION COSINE MATRIX ORIENTING THE B_i FRAME WITH RESPECT TO THE $\bar{G}_i$ FRAME, NORMALLY THROUGH POSITIVE ANGLES FOR RIGHT HAND ROTATIONS IN GOING FROM $\bar{G}_i$ TO THE $\bar{B}_i$ FRAME. THE MATRIX ORIENTING THE $\bar{G}_i$ FRAME WITH RESPECT TO THE $\bar{B}_i$ FRAME (SIGNS OF ANGLES REVERSED) IS

$$ M_{G/B} = M^{*}_{B/G} $$

WHERE ASTERISK DESIGNATES THE TRANSPOSE MATRIX.

THE ELEMENTS ARE DESIGNATED AS

$$ M_{B/G} = (B^{G}_{ij}) , $$

WHICH MAY BE READ AS:

$$ B^{G}_{ij} = \bar{B}_i \cdot \bar{G}_j $$

AND

$$ G^{B}_{ji} = \bar{G}_j \cdot \bar{B}_i $$

CONSEQUENTLY SINCE $M_{B/G} = M^{*}_{G/B}$

$$ (B^{G}_{ij}) = (G^{B}_{ij})^{*} = (G^{B}_{ji}) $$

OR

$$ G^{B}_{ji} = B^{G}_{ij} . $$

UNIT VECTORS

(ALL SETS OF UNIT VECTORS ARE SETS OF MUTUALLY ORTHOGONAL UNIT VECTORS.)

$\bar{A}_1, \bar{A}_2, \bar{A}_3$	AERODYNAMIC FRAME ALONG WHICH AERODYNAMIC FORCES ARE REFERRED $\bar{A}_1 = \bar{W}_1$, THE DRAG AXIS; $\bar{A}_2$ AND $\bar{A}_3$ IN DIRECTIONS OF SIDE AND LIFT FORCE.
$\bar{B}_1, \bar{B}_2, \bar{B}_3$	BODY FIXED FRAME $\bar{B}_1$ ALONG MISSILE LONGITUDINAL AXIS, $\bar{B}_2$ ALONG MISSILE PITCH AXIS.
$\bar{C}_1, \bar{C}_2, \bar{C}_3$	CONTROL AXES
$\bar{E}_1, \bar{E}_2, \bar{E}_3$	EARTH FIXED REFERENCE FRAME $\bar{E}_3$ ALONG EARTH'S POLAR AXIS, $\bar{E}_1$ AND $\bar{E}_2$ LIE IN THE EQUATORIAL PLANE.
$\bar{F}_1, \bar{F}_2, \bar{F}_3$	FIN FIXED FRAME $\bar{F}_1 = \bar{B}_1$, $\bar{F}_2$ AND $\bar{F}_3$ LIE IN THE PLANE CONTAINING THE FIN PANELS, THUS, FOR A MISSILE WITH A CRUCIFORM CONFIGURATION $\bar{F}_2$ AND $\bar{F}_3$ WOULD BE ROLLED OVER SAY 45° FROM THE $\bar{B}_2$ AND $\bar{B}_3$ AXES. (SEE FIGURE .)
	FIXED FRAME FOR GENERAL DISCUSSION $\bar{F}_1$ FRAME CONSIDERED A FIXED FRAME (NOT NECESSARILY WITH RESPECT TO INERTIAL SPACE) AND $\bar{R}_i$ THE ROTATING FRAME.
$\bar{G}_1, \bar{G}_2, \bar{G}_3$	GROUND FRAME $\bar{G}_3$ PERPENDICULAR TO THE LOCAL TANGENT PLANE TO THE EARTH, IF A SPHEROIDAL EARTH $\bar{G}_3$ ALONG THE "PLUMB-BOB" VERTICAL. $\bar{G}_2$ AND $\bar{G}_1$ LIE IN LOCAL TANGENT PLANE TO EARTH.
$\bar{H}_1, \bar{H}_2, \bar{H}_3$	HEIGHT OF MISSILE ABOVE SURFACE OF EARTH MEASURED ALONG $\bar{H}_3 = \bar{G}_3$, H_1 LOCAL EAST, $\bar{H}_2$ LOCAL NORTH.
$\bar{H}_{iT}$	EAST, NORTH AND VERTICAL FRAME AT TARGET.
$\bar{I}_i$	INERTIAL FRAME $\bar{I}_3 = \bar{E}_3$.
$\bar{L}_i$	LAUNCH FRAME
$\bar{M}$	MOVING MISSILE FRAME MOVING WITH RESPECT TO THE FIN FRAME $\bar{F}_1$, $\bar{M}_1 = \bar{F}_1$, $\bar{M}_3$ LIES ALONG THE LINE OF INTERSECTION OF PLANE CONTAINING $\bar{B}_1$ AND THE VELOCITY OF THE MISSILE WITH RESPECT TO THE WIND AND THE PLANE CONTAINING $\bar{B}_2$ AND $\bar{B}_3$ (NORMAL TO LONGITUDINAL BODY AXIS). AERODYNAMIC FORCES ARE OFTEN GIVEN AS CHORD FORCE ALONG $\bar{M}_1$ AND TWO SIDE FORCES ALONG $\bar{M}_2$ AND $\bar{M}_3$.
$\bar{P}_i$	PLATFORM FRAME FIXED TO STABLE PLATFORM.

GYRO FRAME WHEN USED WITH A FREE GYRO TWO OF THE P_i VECTORS LIE IN THE PLANE OF THE ROTOR AND ARE SPINNING WITH THE GYRO ROTOR.

$\bar{R}_i$ — RADAR FRAME FRAME OF REFERENCE FROM WHICH RADAR AZIMUTH AND ELEVATION ANGLES ARE MEASURED.

$\bar{S}_i$ — SENSING AXES (ACCELEROMETER SENSING AXES).

SIGHTLINE FRAME S_1 ALONG RADIUS VECTOR FROM RADAR SITE TO TRACKED OBJECT. (DOUBLE SUBSCRIPTS ARE OFTEN USED, FOR EXAMPLE S_{iM}, S_{iT} MAY REFER TO THE SIGHTLINE FRAME FROM THE OBSERVATION POINT TO THE MISSILE AND TO THE TARGET RESPECTIVELY.)

$\bar{T}_i$ — PATH TANGENT FRAME (SPACE TRAJECTORY).

T_1 TANGENTIAL TO TRAJECTORY.

T_2 PARALLEL TO LOCAL HORIZONTAL.

T_3 LIES IN VERTICAL PLANE TO LOCAL HORIZONTAL.

$\bar{T}_i$ — THRUST FRAME T_i ALONG THRUST FORCE AXIS.

$\bar{T}_i^*$ — RELATIVE PATH TANGENT AS OBSERVED BY A GROUND FIXED OBSERVER, I.E. ${}^E\bar{V}_M = {}^EU_M \ \bar{T}_i^*$.

$\bar{K}_i$ — UNIT VECTORS, $K_i = H_i$ FOR A SPHERICAL EARTH ASSUMPTION, OTHERWISE $\bar{K}_3 \cdot \bar{H}_3 = \cos(\lambda - \lambda^*)$.

EULER ANGLES

ϕ — "ROLL" ANGLE MEASURING ROTATION ABOUT R_{1_J} (THE $\bar{R}_1$ VECTOR AFTER THE FIRST, SECOND OR THIRD ROTATION, DEPENDING ON THE SEQUENCE OF ROTATIONS.

θ — "PITCH" ANGLE MEASURING ROTATION ABOUT R_{2_J} (THE R_2 VECTOR AFTER J^{TH} ROTATION).

ψ — "YAW" ANGLE MEASURING ROTATION ABOUT THE R_{3_J} (THE $\bar{R}_3$ VECTOR AFTER J^{TH} ROTATION).

$\phi_{B/G}$ — $\theta_{B/G}$, $\psi_{B/G}$ - EULER ANGLES DEFINING THE ORIENTATION OF THE BODY FRAME B_i WITH RESPECT TO THE GROUND FRAME G_i. THE EULER SEQUENCE MUST ALSO BE SPECIFIED, FOR EXAMPLE YAW, PITCH, ROLL OR

$\bar{B}_i = M_R\ (\phi_{B/G})\ M_P\ (\theta_{B/G})\ M_Y\ (\psi_{B/G})\ G_i$. IN GENERAL, THE SUBSCRIPTS MEAN

$B/_G$ Body frame with respect to ground frame.

$B/_R$ Body frame with respect to the radar frame.

$G/_H$ $\overline{G}_i$ frame with respect to $\overline{H}_i$ frame.

(Ψ, θ, ϕ) Ordering implies sequence yaw, pitch, roll etc.

Some angles commonly used are:

β, α, angles of attack in yaw (sideslip) and pitch; or Euler angles orienting the body frame $\overline{B}_i$ with respect to the wind frame $\overline{W}_i$ for the sequence yaw, pitch, zero roll, or vice versa.

$$\text{i.e.}\quad \beta = \Psi\, B/_W$$
$$\alpha = \theta\, B/_W$$

α_T Total angle of attack, angle between $\overline{B}_1$ and $\overline{W}_1$ vector.

ϕ_A Roll angle associated with aerodynamic wind tunnel data.

The orientation of the moving missile axes $\overline{M}_i$ with respect to the body fixed fin axes $\overline{F}_i$ is given in terms of the roll matrix

$$\overline{M}_1 = M_R\ (\phi_A)\ \overline{F}_i\ .$$

The orientation of the aerodynamic axes $\overline{A}_i$ with respect to the body fixed fin axes $\overline{F}_i$ is given in terms of a roll and pitch matrix.

$$\overline{A}_1 = M_P\ (\alpha_T)\ M_R\ (\phi_A)\ \overline{F}_i\ .$$

Euler angles orienting the missile velocity vector with respect to a local horizontal frame say $\overline{H}_i$ through a yaw and a pitch (zero roll) matrix.

$$\text{i.e.}\quad \Psi_V = \Psi\, V/_H,\ \gamma = \theta\, V/_H\ .$$

A, E Azimuth and elevation angles of missile, target etc., depending on subscripts. For example, the orientation of the missile - sightline frame S_{iM} with respect to the missile track frame $\overline{R}_{iM}$ through a missile yaw and a pitch is:

$$\overline{S}_{iM} = M_P\ (\theta\ S/_R)\ M_Y\ (\Psi\, S/_R)\ \overline{R}_{iM}$$

where $$A_M = \pm\Psi\, S/_R,\ E_M = \pm\,\theta\ S/_R$$

(sign depending on way defined for particular system).

G_O, G_I, G_S GIMBAL PICK-OFF ANGLES ORIENTING A GYRO OR PLATFORM REFERENCE FRAME WITH RESPECT TO MISSILE BODY AXES. THE SUBSCRIPTS O, I, S DESIGNATE OUTER, INNER AND SPINNER (FOR A TWO DEGREE OF FREEDOM FREE GYRO, THE SPINNER ANGLE IS NOT UTILIZED AS A PICK-OFF).

P, Q, R ANGULAR VELOCITY COMPONENTS ABOUT THE THREE MUTUALLY ORTHOGONAL AXES OF A REFERENCE FRAME (NUMBER ONE AXIS, NUMBER TWO AXIS, AND NUMBER THREE AXIS RESPECTIVELY).

P_B, Q_B, R_B INERTIAL ANGULAR VELOCITY COMPONENTS OF THE MISSILE BODY (THE ABSENCE OF A SECOND SUBSCRIPT IMPLIES THE INERTIAL FRAME OR ORIGIN). THE ABSENCE OF A SUPERSCRIPT MEANS THAT THE COMPONENTS THE BODY FRAME RATES ARE MEASURED IS THE SAME (BODY) FRAME.

$P^G_{B/R}$ $Q^G_{B/R}$ $R^G_{B/R}$ SAME DESIGNATION AS ABOVE EXCEPT THAT THE SECOND SUBSCRIPT MEANS THE ANGULAR VELOCITY OF THE BODY FRAME IS MEASURED WITH RESPECT TO THE RADAR AXES (NOT NECESSARILY AN INERTIAL FRAME). THE SUPERSCRIPT MEANS THAT THE ANGULAR VELOCITY VECTOR OF THE $\overline{B}_i$ FRAME WITH RESPECT TO THE $\overline{R}_i$ FRAME IS RESOLVED INTO COMPONENTS ALONG THE $\overline{G}_i$ AXIS DIRECTIONS. ETC., FOR OTHER SUB AND SUPERSCRIPTS. SUBSCRIPT:

$B/_G$ BODY FRAME WITH RESPECT TO GROUND.

$B/_E$ BODY FRAME WITH RESPECT TO EARTH FRAME.

$G/_E$ $\overline{G}_i$ FRAME WITH RESPECT TO THE $\overline{E}_i$ FRAME.

POSITION VECTORS

$\overline{R}_M$ POSITION VECTOR OF THE MISSILE CENTER OF MASS WITH RESPECT TO THE INERTIAL ORIGIN AT THE CENTER OF THE EARTH.

$\overline{R}_{M/R}$ POSITION VECTOR OF MISSILE C. G. WITH RESPECT TO A RADAR SITE ORIGIN.

$\overline{R}_{M/T}$ POSITION VECTOR OF THE MISSILE C. G. WITH RESPECT TO THE TARGET.

$\overline{R}_T$ POSITION VECTOR OF TARGET WITH RESPECT TO THE ORIGIN AT THE CENTER OF THE EARTH.

$\overline{R}_R$ POSITION VECTOR OF THE RADAR ORIGIN WITH RESPECT TO THE CENTER OF THE EARTH.

$\overline{R}_A$ POSITION VECTOR OF ACCELEROMETER WITH RESPECT TO INERTIAL ORIGIN AT CENTER OF EARTH.

RECTANGULAR COORDINATES

$(X^R_{M/T}, Y^R_{M/T}, Z^R_{M/T})$ RECTANGULAR COORDINATES OF MISSILE WITH RESPECT TO THE TARGET COMPONENTS MEASURED IN THE INSTANTANEOUS $\overline{R}_1$, $\overline{R}_2$, AND $\overline{R}_3$ DIRECTIONS RESPECTIVELY.

THE FOLLOWING GENERAL SCHEME IS USED.

(1) X, Y, Z DESIGNATE FIRST, SECOND AND THIRD COORDINATES RESPECTIVELY.

(2) SUBSCRIPTS DESIGNATE THE COORDINATE OF POINT M WITH RESPECT TO POINT R.

(3) THE SUPERSCRIPT DESIGNATES THE REFERENCE FRAME TAKEN AS A BASIS, I.E. REFERENCE FRAME IN WHICH THE COORDINATES ARE MEASURED.

VELOCITY VECTORS

$\frac{d\bar{R}_M}{dt}$ VELOCITY OF MISSILE C. G. WITH RESPECT TO THE INERTIAL ORIGIN O AT CENTER OF EARTH.

$\frac{d\bar{R}_{M/T}}{dt}$ VELOCITY OF POINT M (MISSILE) WITH RESPECT TO THE TARGET.

$\frac{d\bar{R}_{M/W}}{dt}$ VELOCITY OF THE MISSILE WITH RESPECT TO THE WIND.

$\frac{d\bar{R}_{M/R}}{dt}$ VELOCITY OF MISSILE WITH RESPECT TO POINT R (RADAR SITE).

${}^{E}U_M$, ${}^{E}V_M$, ${}^{E}W_M$ THE THREE COMPONENTS OF THE VELOCITY OF THE MISSILE (POINT M) WITH RESPECT TO POINT O (INERTIAL POINT) $\frac{{}^{E}\partial R_M}{\partial t}$ AS OBSERVED FROM THE EARTH FIXED FRAME E_i AND COMPONENTS TAKEN IN THE E_i DIRECTION.

${}^{E}U_M^B$, ${}^{E}V_M^B$, ${}^{E}W_M^B$ THE THREE COMPONENTS OF VELOCITY OF POINT M WITH RESPECT TO POINT O AS OBSERVED FROM THE EARTH FIXED $\bar{E}_i$ FRAME AND COMPONENTS TAKEN IN THE $\bar{B}_i$ DIRECTIONS.

U_M^B, V_M^B, W_M^B THE COMPONENTS OF THE VELOCITY OF POINT M WITH RESPECT TO POINT O AS OBSERVED FROM AN INERTIAL FRAME, AND COMPONENTS IN THE $\bar{B}_i$ FRAME.

$U^B_{M/W}$, $V^B_{M/W}$, $W^B_{M/W}$ THREE COMPONENTS OF THE VELOCITY OF THE MISSILE WITH RESPECT TO THE LOCAL AIR AS OBSERVED FROM AN INERTIAL FRAME, COMPONENTS IN THE B_i DIRECTIONS.

IN GENERAL

$${}^{O}U^{C}_{P/Q}$$

DESIGNATES

(1) U IMPLIES THE FIRST COMPONENT.

(2) THE SUBSCRIPTS DESIGNATE THE VELOCITY OF POINT P WITH RESPECT TO WHAT POINT Q.

(3) THE LEFT SUPERSCRIPT O DESIGNATES THE REFERENCE FRAME FROM WHICH THE OBSERVATION IS MADE.

(4) THE RIGHT SUPERSCRIPT DESIGNATES THE REFERENCE FRAME IN WHICH THE COMPONENTS ARE EXPRESSED I.E. THE $\overline{C}_i$ FRAME.

DIRECTION COSINE MATRICES

$M_{B/G}$ THE DIRECTION COSINE MATRIX ORIENTING THE BODY FRAME $\overline{B}_i$ WITH RESPECT TO THE GROUND FRAME $\overline{G}_i$.

$M_{G/B}$ THE ORIENTATION OF THE $\overline{G}_i$ FRAME WITH RESPECT TO THE $\overline{B}_i$ FRAME.

$M^*_{G/B}$ ASTERISK DESIGNATES AS A SUPERSCRIPT ON A MATRIX THE TRANSPOSE. $M^*_{G/B} = M_{B/G}$ FOR ALL DIRECTION COSINE MATRICES USED HERE.

$M_{B/G} = (B^G_{ij})$ THE ELEMENTS OF THE MATRIX ORIENTING THE $\overline{B}_i$ REFERENCE FRAME WITH RESPECT TO THE $\overline{G}_i$ FRAME. OTHER SUBSCRIPTS AS NEEDED.

FORCES

$\overline{F}_A$ RESULTANT AERODYNAMIC FORCE.

$F^B_{A_1}$, $F^B_{A_2}$, $F^B_{A_3}$ MAGNITUDES OF THE THREE COMPONENTS OF THE RESULTANT AERODYNAMIC FORCE ACTING ON THE MISSILE EXPRESSED IN THE $\overline{B}_i$ REFERENCE FRAME.

THE SUPERSCRIPT DESIGNATES THE REFERENCE FRAME TO WHICH COMPONENTS ARE REFERRED, THE OTHER SUPERSCRIPTS ARE:

$F^W_{A_i}$, $F^M_{A_i}$, $F^A_{A_i}$ DESIGNATING THE THREE AERODYNAMIC FORCES $(i = 1, 2, 3)$ REFERRED TO THE WIND AXES $\overline{W}_i$, MOVING AXES $\overline{M}_i$, AERODYNAMIC AXES $\overline{A}_i$, (SEE FIG. 12.5).

$F^A_{A_1} = F^W_{A_1} = D$ D COMMONLY CALLED THE DRAG FORCE AND L A LIFT FORCE.

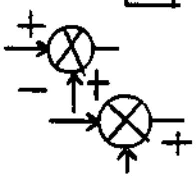
$\overline{F}_T$ THRUST FORCE.

F^T_T MAGNITUDE OF THRUST FORCE ACTING ALONG THRUST AXIS

$F^{B}_{T_i}$	MAGNITUDES OF THE COMPONENTS OF THE THRUST VECTOR ACTING ALONG BODY AXES.
$\bar{F}_{N/G}$	NEWTONIAN MASS ATTRACTION GRAVITATIONAL FORCE.
$\bar{F}_{G_E}$	EFFECTIVE GRAVITATIONAL FORCE.
$F^{B}_{(GE)_i}$	MAGNITUDES OF THE EFFECTIVE GRAVITATIONAL FORCE IN THE $\bar{B}_i$ REFERENCE FRAME.
O	INERTIAL ORIGIN AT CENTER OF EARTH, GEOCENTRIC EARTH'S CENTER.
O'	GEODETIC EARTH CENTER, MOVING POINT ON POLAR AXIS AT WHICH THE NORMAL TO THE TANGENT PLANE OF THE SPHEROID INTERSECTS THE POLAR AXIS.
R^{*}_{E}	GEOCENTRIC EARTH'S RADIUS.
R_E	LENGTH OF THE GREAT NORMAL OF THE SPHEROID (THE GEODETIC RADIUS).
A	EQUATIONAL RADIUS OF EARTH (SEMI - MAJOR AXIS).
B	SEMI - MINOR AXIS OR POLAR RADIUS OF EARTH.
E	ECCENTRICITY OF THE EARTH.
$A = (1-E^2)$	
F	MEAN POLAR FLATTENING.
L^B_A, M^B_A, N^B_A	MAGNITUDE OF THE EXTERNAL MOMENTS ABOUT THE BODY AXES B_1, B_2, B_3 RESPECTIVELY.
L_T, M_T, N_T	THRUST MOMENTS (DUE TO THRUST MALALIGNMENT).
M_A	MACH NUMBER.
M	MISSILE MASS.
Q	DYNAMIC PRESSURE.
B	WIND SPAN.
c_M	MEAN AERODYNAMIC CHORD.
C_L	DIMENSIONLESS LEFT COEFFICIENT.

C_D	DIMENSIONLESS DRAG COEFFICIENT.
C_S	DIMENSIONLESS SIDE FORCE COEFFICIENT.
C_E	DIMENSIONLESS ROLLING MOMENT COEFFICIENT.
C_M	DIMENSIONLESS PITCHING MOMENT COEFFICIENT.
C_N	DIMENSIONLESS YAWING MOMENT COEFFICIENT.

SYMBOLS

$\boxed{S}$	DIFFERENTIATOR.
$\boxed{1/S}$	INTEGRATOR.
$\boxed{X}$	MULTIPLIER.
	DIFFERENTIAL (SUBTRACTION).
$\boxed{\Sigma}$	SUMMATION.
$\frac{D}{DT} = {}^{B}\frac{\partial}{\partial T} + \bar{\omega}_B \times$	OPERATOR ON A VECTOR.
S	SINE.
C	COSINE.

APPENDIX A

ELEMENTARY VECTOR THEORY

The few elementary vector properties used in the derivations in this report can generally be found in the first few sections of any vector analysis text. A formal course in vector analysis is not a prerequisite to the understanding of this report. A brief intuitive feeling for these elementary properties is attempted in this appendix for those busy, interested persons who never have time to dig into a vector text.

SCALAR OR DOT PRODUCT

Consider a force field $\bar{F}$ and a space curve C along which a particle is moving. In freshman physics the increment of work dw (a scalar quantity) is defined as the product of the component of the force in the direction of displacement and the scalar magnitude of displacement.

$$dw = (F \cos \theta)ds \qquad \text{(A-1)}$$

Since the force and displacement have magnitude and direction, they are the vector quantities shown in Fig. (A-1). Eq(A-1)

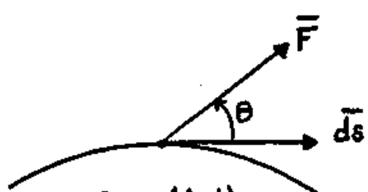


Fig (A-1)

can be written as

$$dw = \bar{F} \cdot \bar{ds} = |\bar{F}|\,|\bar{ds}| \cos \theta \qquad \text{(A-2)}$$

where $|\ |$ means scalar magnitude.

Obviously the scalar dot product is a scalar quantity and is the scalar product of a vector and the projection of a second vector onto it, thus

$$\bar{X} \cdot \bar{Y} = |\bar{X}|\,|\bar{Y}| \cos(\bar{X},\bar{Y}) \ , \qquad \text{(A-3)}$$

and

$$\bar{X} \cdot \bar{X} = |\bar{X}|\,|\bar{X}| \cos 0^\circ = |\bar{X}|^2 \ .$$

As a geometrical example consider the position vector $\bar{R}$ of a point in terms of its rectangular components. By Fig. (A-2)

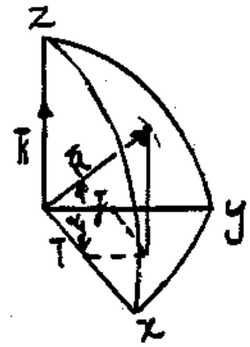


Fig. (A-2)

$$\bar{R} = x\bar{i} + y\bar{j} + z\bar{k} \tag{A-4}$$

Dotting E_1 (A-4) by $\bar{R}$,

$$\bar{R}\cdot\bar{R} = |\bar{R}|^2 = (k\bar{i} + y\bar{j} + z\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$|R|^2 = x^2 + y^2 + z^2,$$

$$\text{since: } \bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$$

$$\bar{i}\cdot\bar{j} = \bar{i}\cdot\bar{k} = \bar{k}\cdot\bar{j} = 0 \tag{A-5}$$

If $|\bar{R}|$ is a constant, then $\bar{R}$ is the position vector of the points on a sphere.

Note also that

$$\bar{R}\cdot\bar{i} = k = R \cos\alpha_1. \tag{A-6}$$

VECTOR, OR CROSS, PRODUCT

Consider the motion of a particle moving on a circle in the X, Y Plane of Fig. (A-3).

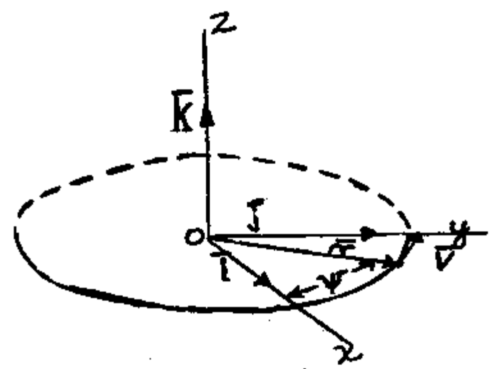


Fig (A-3)

where the $\bar{k}$ axis is the axis of rotation. The tangential speed $\bar{V}$ is

$$V = \dot{\psi} r = \omega r \qquad \text{(A-7)}$$

where ω is the angular speed $\dot{\psi}$. The angle ψ is not a vector; however $\dot{\psi}\bar{k} = \bar{\omega}$ is a vector since angular velocity obeys vector laws. The angular velocity of the particle is a vector along the axis of rotation whose magnitude is equal to the scalar angular speed $\dot{\psi}$ (positive in positive $\bar{k}$ direction for counter clockwise rotation as shown in Fig. (A-3)).

Assume that the particle is fixed to the earth at a constant latitude and rotating with the earth as shown in Fig. (A-4).

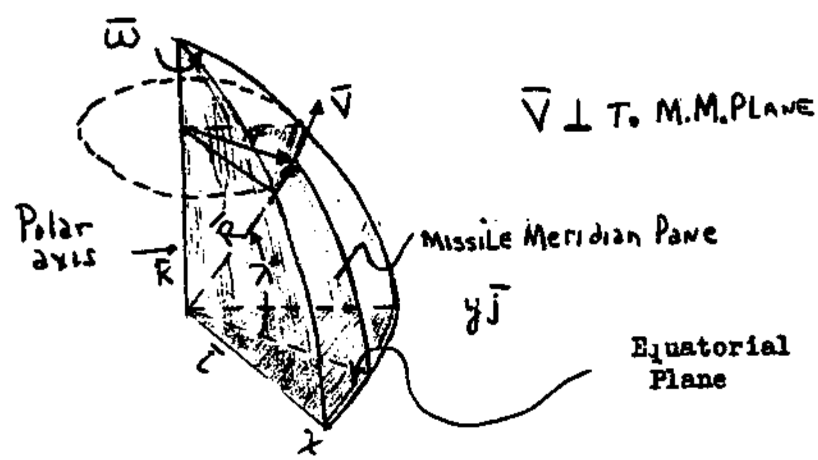


Fig. (A-4)

Scalarwise,

$$r = R \sin (90-\lambda^{*}) = R \cos \lambda^{*} \quad . \tag{A-8}$$

Vectorwise,

$$\bar{R} = z\,\bar{k} + r\bar{r}_1 \tag{A-9}$$

where $\bar{r}_1$ is a unit vector in direction of r.

Dotting (A-9) by $\bar{r}_1$

$$\bar{R} \cdot \bar{r}_1 = |\bar{R}| \cos \lambda^{*} = r \tag{A-10}$$

Utilizing Eq. (A-8) in (A-7)

$$V = \omega r = \omega R \sin (90-\lambda^{*}). \tag{A-11}$$

Abrreviating Eq(A-11) in vector form

$$\bar{V} = \bar{\omega} X \bar{R} = |\bar{\omega}|\,|\bar{R}| \sin (\bar{\omega},\bar{R})\,\bar{T}. \tag{A-12}$$

Where $\bar{T}$ is a unit tangent vector.

From the above it is seen that the tangential velocity is tangential to the circle on the earth at a constant latitude, and that this vector $\bar{V}_{\perp}$ is perpendicular to the plane containing the $\bar{\omega}$ and the position vector $\bar{R}$. At the equator $\lambda^{*} = 0$, and Eq(A-11) gives the speed of a particle fixed to the surface of the earth at the equator, i.e. $V = \omega r$ at equator.

In general, for two vectors $\bar{A}$ and $\bar{B}$ of Fig. (A-5), the cross product $\bar{A} X \bar{B}$ is

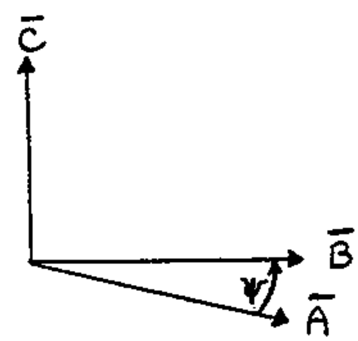


Fig. (A-5)

$$\bar{A} X \bar{B} = \bar{C} \tag{A-13}$$

and $$|\bar{A} X \bar{B}| = |\bar{A}|\,|\bar{B}| \sin (\bar{A},\bar{B}) \tag{A-14}$$

For the right handed mutually orthogonal reference frame established by the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$, of Fig. (A-3), one obtains:

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = \bar{k} \qquad \bar{j} \times \bar{i} = -\bar{k}$$

$$\bar{j} \times \bar{k} = \bar{i} \qquad \bar{k} \times \bar{j} = -\bar{i}$$

$$\bar{k} \times \bar{i} = \bar{j} \qquad \bar{i} \times \bar{k} = -\bar{j} .$$

The cross product of the two vectors $\bar{A}$ and $\bar{B}$,

where $\bar{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{B} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

is

$$\bar{A} \times \bar{B} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \times (b_1\bar{i} + b_2\bar{j} + b_3\bar{k})$$

$$\bar{A} \times \bar{B} = (a_2b_3 - b_2a_3)\bar{i} + (a_3b_1 - a_1b_3)\bar{j} + (a_1b_2 - b_1a_2)\bar{k}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(A-15)}$$

TRIPLE SCALAR PRODUCT

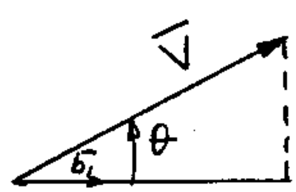
Now consider the velocity vector of Fig. (A-4) which, in terms of $\bar{\omega}$ and $\bar{R}$, is given by Eq (A-13) as

$$\bar{V} = \bar{\omega} \times \bar{R}. \qquad \text{(A-16)}$$

Suppose the instantaneous component of $\bar{V}$ in the direction of $\bar{b_1}$ is desired, where $\bar{b_1}$ is the unit vector along the missile longitudinal axis, then

$$\bar{V} \cdot \bar{b_1} = |\bar{V}||\bar{b_1}| \cos \theta \qquad \text{(A-17)}$$

as shown in sketch.



$|\bar{V}| \cos \theta \, \bar{b_1}$

By Eq (A-16) and (A-17)

$$\bar{b}_1 \cdot \bar{V} = \bar{b}_1 \cdot (\bar{\omega} \times \bar{R}), \qquad \text{(A-18)}$$

which is the triple scalar product, a scalar quantity.

TRIPLE VECTOR PRODUCT

Consider now the freshman physics concept of angular acceleration for the discussion of Fig. (A-3). If the angular speed is a constant, then the magnitude of the tangential velocity vector is a constant and the only acceleration of the particle is along the negative of the radius vector (toward the polar axis) as shown in Fig (A-6). The magnitude of the so-called centripetal acceleration is given as

$$A = \frac{V^2}{r} \qquad \text{(A-19)}$$

By Eq (A-7) and (A-19)

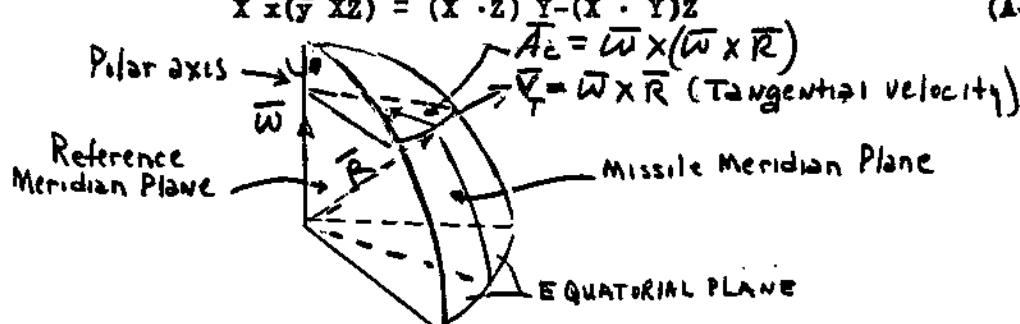
$$A = V\left(\frac{V}{r}\right) = \omega r\left(\frac{V}{r}\right) = \omega V \ . \qquad \text{(A-20)}$$

In vector form, utilizing Eq (A-12)

$$\bar{A}_c = \bar{\omega} \times \bar{V} = \bar{\omega} \times (\bar{\omega} \times \bar{R}). \qquad \text{(A-21)}$$

The term $\bar{\omega} \times (\bar{\omega} \times R)$ is the triple vector product, and is a vector normal to the $\bar{\omega}$ vector and the $\bar{\omega} \times \bar{R} = \bar{V}$ vector. Thus it is seen that at the equator this acceleration vector is directed toward the origin, while at latitude λ^*, it is centrally directed toward the polar axis and lies in the plane of the circle tangent to the velocity vector. This vector is shown in any vector text book to be

$$\bar{X} \times (\bar{Y} \times \bar{Z}) = (\bar{X} \cdot \bar{Z})\,\bar{Y} - (\bar{X} \cdot \bar{Y})\,\bar{Z} \qquad \text{(A-22)}$$



FIG(A-6)

DIFFERENTIATION OF VECTORS

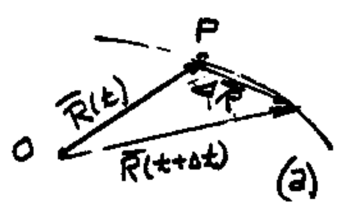
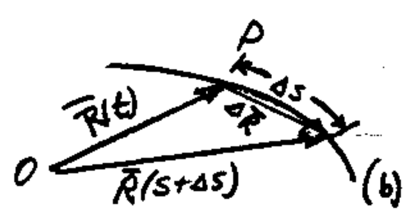


Fig.(A-7)



Consider the time rate of change of the position vector $\overline{R}$ when $\overline{R}(t)$ is a vector function of time. By Fig. (A-7)

$$\overline{R}(t) + \Delta\overline{R} = \overline{R}(t + \Delta t)$$

$$\Delta\overline{R} = \overline{R}(t + \Delta t) - \overline{R}(t)$$

$$\overline{V} \equiv \frac{d\overline{R}}{dt} = \lim_{\Delta t \to 0} \frac{\overline{\Delta R}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{R}(t + \Delta t) - \overline{R}(t)}{\Delta t}. \qquad \text{(A-23)}$$

$\overline{R}$ may also be expressed as a function of arc length S, $\overline{R}(s)$, then

$$\overline{R}(s) + \Delta\overline{R} = \overline{R}(s + \Delta s)$$

$$\Delta\overline{R} = \overline{R}(s + \Delta s) - \overline{R}(s)$$

$$\frac{d\overline{R}}{ds} = \lim_{\Delta S \to 0} \frac{\Delta\overline{R}}{\Delta S} = \lim_{\Delta S \to 0} \frac{\overline{R}(s + \Delta s) - \overline{R}(s)}{\Delta S} = \left|\lim_{\Delta S \to 0} \frac{\Delta\overline{\ }}{\Delta S}\right| \overline{T} = \overline{T}. \qquad \text{(A-24)}$$

where T is the unit tangent vector,

$$\text{since} \left|\lim_{\Delta S \to 0} \frac{\Delta\overline{R}}{\Delta S}\right| = 1 .$$

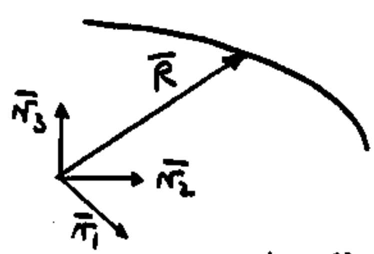
from Eq (A-23) and (A-24)

$$\overline{V} \equiv \frac{d\overline{R}}{dt} = \frac{d\overline{R}}{ds}\frac{ds}{dt} = \overline{T}\,\frac{ds}{dt} = V\overline{T} \qquad \text{(A-25)}$$

where $\frac{ds}{dt} = V = |\overline{V}|$ is the scalar speed along the path.

From the foregoing equations it is obvious that the time rate of change of position vector is the velocity vector and is tangential to the path.

If now $\overline{R}$ is expressed in the $\overline{r_i}$ reference frame, from Fig. (A-8)



$$\overline{R} = X_1\overline{r_1} + X_2\overline{r_2} + X_3\overline{r_3} \qquad \text{(A-26)}$$

Taking the derivative of equation (A-26), and remembering that the $\overline{r_i}$ reference frame is moving,

$$\frac{d\overline{R}}{dt} = \dot{X}_1\overline{r_1} + X_1\dot{\overline{r_1}} + \dot{X}_2\overline{r_2} + X_2\dot{\overline{r_2}} + \dot{X}_3\overline{r_3} + X_3\dot{\overline{r_3}}$$

$$\dot{\bar{R}} = (\dot{X}_1\bar{r}_1 + \dot{X}_2\bar{r}_2 + \dot{X}_3\bar{r}_3) + (X_1\dot{\bar{r}}_1 + X_2\dot{\bar{r}}_2 + X_3\dot{\bar{r}}_3)$$

$$\dot{\bar{R}} = \frac{\partial \bar{R}}{\partial t} + (X_1\dot{\bar{r}}_1 + X_2\dot{\bar{r}}_2 + X_3\dot{\bar{r}}_3) \quad . \tag{A-27}$$

The term $\frac{\partial \bar{R}}{\partial t}$ is called the apparent velocity and is equal to the velocity $\frac{d\bar{R}}{dt}$ when $\bar{r}_i$ is an inertial reference frame.

Consider the unit vector $\bar{r}_1(t)$ and $\bar{r}_1(t + \Delta t)$ at a time Δt later, vectorally

FIG.

$$\Delta\bar{r}_1 = \bar{r}_1(t + \Delta t) - \bar{r}_1(t)$$

$$\lim_{\Delta t \to 0} \frac{\Delta\bar{r}_1}{\Delta t} \equiv \dot{\bar{r}}_1 = 0\bar{r}_1 + b_1\bar{r}_2 + C_1\bar{r}_3 = \dot{\mu}\bar{n} \tag{A-28}$$

t 0 t

Eq (A-28) states that $\dot{\bar{r}}_1$ is a vector normal to the $\bar{r}_1$ vector as is obvious from the Fig. (A-9), for as $\Delta t \to 0$, the incremental vector $\Delta\bar{r}_1$ approaches the tangent to a unit circle ($\dot{\mu}$ is an angular rate with respect to inertial space). In a similar manner $\dot{\bar{r}}_2$ and $\dot{\bar{r}}_3$ can be expressed as components in the $\bar{r}_i$ frame

$$\begin{aligned} \dot{\bar{r}}_1 &= 0\bar{r}_1 + a\bar{r}_2 + b\bar{r}_3 \\ \dot{\bar{r}}_2 &= c\bar{r}_1 + 0\bar{r}_2 + d\bar{r}_3 \\ \dot{\bar{r}}_3 &= e\bar{r}_1 + f\bar{r}_2 + 0\bar{r}_3 \end{aligned} \tag{A-29}$$

also

$$\bar{r}_1 = \bar{r}_2 \times \bar{r}_3 \tag{A-30}$$

and the derivative of Eq (A-30) is given by Eq (   ) as

$$\dot{\bar{r}}_1 = \bar{r}_2 \times \dot{\bar{r}}_3 + \dot{\bar{r}}_2 \times \bar{r}_3 \tag{A-31}$$

$$= \bar{r}_2 X(e\bar{r}_1 + f\bar{r}_2) + (c\bar{r}_1 + d\bar{r}_3) X \bar{r}_3$$

$$\dot{\bar{r}}_1 = -e\bar{r}_3 - c\bar{r}_2 \quad . \tag{A-32}$$

By Eq (A-29) and (A-32)

$$\begin{aligned} a &= -c \\ b &= -e \quad . \end{aligned} \tag{A-33}$$

In a similar manner

$$\bar{r}_3 = \bar{r}_1 X \bar{r}_2 \tag{A-34}$$

$$\dot{\bar{r}}_3 = \bar{r}_1 X \dot{\bar{r}}_2 + \dot{\bar{r}}_1 X \bar{r}_2$$

Using (A-33) and (A-29) in (A-34)

$$\dot{\bar{r}}_3 = \bar{r}_1 X(-a\bar{r}_1 + d\bar{r}_3) + (a\bar{r}_2 - e\bar{r}_3) X \bar{r}_2$$

$$\dot{\overline{r_3}} = -d\overline{r_2} + e\overline{r_1} \qquad (A\text{-}35)$$

By (A-35) and (A-29)

$$f = -d \quad . \qquad (A\text{-}36)$$

Using E_1 (A-33) and (A-36) in (A-29) one obtains

$$\begin{aligned}\dot{\overline{r_1}} &= a\overline{r_2} - e\overline{r_3}\\ \dot{\overline{r_2}} &= -a\overline{r_1} + d\overline{r_3}\\ \dot{\overline{r_3}} &= +e\overline{r_1} - d\overline{r_2} \quad .\end{aligned} \qquad (A\text{-}37)$$

Using E_1 (A-37) in (A-27) and $b = -e$ by (A-33)

$$\frac{d\overline{R}}{dt} = \frac{\partial \overline{R}}{\partial t} + x_1(a\overline{r_2} - e\overline{r_3}) + x_2(-a\overline{r_1} + d\overline{r_3}) + x_3(e\overline{r_1} - d\overline{r_2})$$

$$= \frac{\partial \overline{R}}{\partial t} + \overline{r_1}(x_3 e - x_2 a) + \overline{r_2}(x_1 a - x_3 d) + \overline{r_3}(x_2 d - x_1 e)$$

$$\frac{d\overline{R}}{dt} = \frac{\partial \overline{R}}{\partial t} + \begin{vmatrix} \overline{r_1} & \overline{r_2} & \overline{r_3} \\ d & e & a \\ x_1 & x_2 & x_3 \end{vmatrix} = \frac{\partial \overline{R}}{\partial t} + \overline{\omega}_R \times \overline{R} \ . \qquad (A\text{-}38)$$

Eq (A-34) states that the total time derivative of $\overline{R}$ is equal to an apparent velocity plus a component due to the rotation of the reference frame with respect to inertial space.

$$\overline{W}_R = d\,\overline{r}_1 + e\,\overline{r}_2 + a\,\overline{r}_2 + a\,\overline{r}_3 = \overline{P}_r\,\overline{r}_1 + Q_r\,\overline{r}_2 + R_r\,\overline{r}_3 \qquad (A\text{-}39)$$

As a physical example consider the $\overline{r_i}$ reference frame fixed and rotating at the center of the earth, i.e. $\overline{r_i} = \overline{e_i}$.

If $\overline{R} = x_1^e\,\overline{e_1} + x_2^e\,\overline{e_2} + x_3^e\,\overline{e_3}$ and x_i^e ($i = 1\ 2.3$) are constants, i.e. the point P is fixed on the surface of the earth,

then $\dfrac{\partial \overline{R}}{\partial t} = \overline{0}$

and $\dfrac{d\overline{R}}{dt} = \overline{V} = \overline{\omega_e} \times \overline{R}$.

The magnitude of $\overline{V}$ is

$$|\overline{V}| = \omega_e R \sin(90 - \lambda^*) = \omega_e R \cos \lambda^* .$$

at the equator

$$|\overline{V}| = \omega_e R,$$

a person standing on the earth at the equator has a velocity tangential to the circle generated due to the earths' rotation (ignoring translation of earth).

DIFFERENTIATION RULES

A summary of the differentiation rules used in this paper are given here. If $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ are vector functions of t, then

$$\frac{d}{dt}(\overline{X} \pm \overline{y}) = \frac{d\overline{X}}{dt} + \frac{d\overline{y}}{dt} \tag{A-40}$$

$$\frac{d}{dt}\left\{r(t)\overline{X}\right\} = \frac{dr}{dt}\,\overline{X} + r\,\frac{d\overline{X}}{dt} \tag{A-41}$$

$$\frac{d}{dt}(\overline{X} \cdot \overline{y}) = \frac{d\overline{X}}{dt} \cdot \overline{y} + \overline{X} \cdot \frac{d\overline{y}}{dt} \tag{A-42}$$

$$\frac{d}{dt}(\overline{X} x \overline{y}) = \frac{d\overline{X}}{dt} x\, \overline{y} + \overline{X}\, x\, \frac{d\overline{y}}{dt} \tag{A-43}$$

where r(t) of Eq(A-36) is a scalar. Based on the concept of the derivative of a scalar function f(x) as shown in Fig. (A-8)

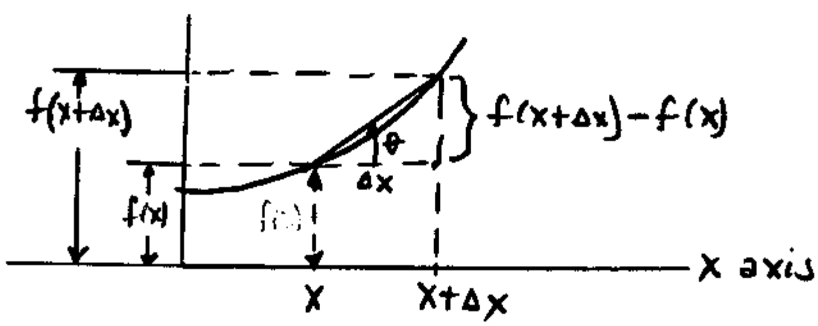


Fig. (A-8)

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \tan\theta \tag{A-44}$$

the demonstration of Eqs (A-42) and (A-43) can be shown as follows:

$$\frac{d}{dt}(\overline{X}\cdot\overline{Y}) = \lim_{\Delta t \to 0} \frac{(\overline{X}+\Delta\overline{X})\cdot(\overline{Y}+\Delta\overline{Y}) - \overline{X}\cdot\overline{Y}}{\Delta t}$$

$$= \frac{\overline{X}\cdot\overline{Y}+\overline{X}\cdot\Delta\overline{Y} + \Delta\overline{X}\cdot\overline{Y} + \Delta\overline{X}\cdot\Delta\overline{Y} - \overline{X}\cdot\overline{Y}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\{ \frac{\Delta\overline{X}}{\Delta t}\cdot\overline{Y} + \overline{X}\cdot\frac{\Delta\overline{Y}}{\Delta t} + \Delta\overline{X}\cdot\frac{\Delta\overline{Y}}{\Delta t} \right.$$

$$\frac{d(\overline{X}\cdot\overline{Y})}{dt} = \dot{\overline{X}}\cdot\overline{Y} + \overline{X}\cdot\dot{\overline{Y}},$$

which is Eq. (A-42) . Eq (A-43) is shown in a similar manner.

APPENDIX B

GENERAL CONSIDERATIONS ON TRANSFORMATIONS

In general, for flight simulation and system synthesis, there are four useful methods of obtaining the direction cosine matrix between two orthonormal reference frames.

(1) The direction cosines can be obtained from the solution of a system of nine first order differential equations in the direction cosines (only three of which are independent). The orthogonal components of inertial angular velocity for both reference frames must be known.

(2) The direction cosines can be obtained as trigonometric functions of Euler angles orienting the one reference frame with respect to the second reference frame. The Euler angles are obtained from the simultaneous solution of a system of three first order differential equations in the Euler angles. The orthogonal components of inertial angular velocity for both reference frames must be known.

(3) The direction cosines between two reference frames can be obtained as the product of two direction cosine matrices when the direction cosine matrices between each of the two reference frames and a common third reference frame are known.

For example if

$$\bar{r}_i = M_{F/r}\,\bar{F}_i$$

and

$$\bar{F}_i = M_{f/F}\,\bar{f}_i$$

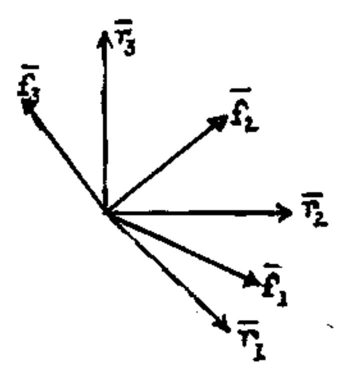
then

$$\bar{r}_i = M_{F/r}\,M_{f/F}\,\bar{f}_i = M\,\bar{f}_i$$

(4) Some of the direction cosines can be obtained from the known rectangular components of a vector.

Direction cosines

The motion of the $\bar{r}_i$ frame with respect to the $\bar{f}_i$ frame, in general, will consist of translational motion and rotational motion. Kinematically, these two motions may be considered separately. If the origin of the $\bar{r}_i$ reference frame is translated a scalar distance x in the instantaneous $\bar{f}_1$ direction, a distance y in the $\bar{f}_2$ direction and finally translated a distance z in the $\bar{f}_3$ direction, the origins of the two frames may be made to coincide. Since the directions of the $\bar{r}_i$ vectors during these three linear translations are not changed, the angular orientations of the two frames have not changed, thus the super-position of origins eases the visualization.



Consider the orientation of one reference frame $\overline{r}_i$ (i = 1,2,3) with respect to a second reference frame $\overline{f}_i$ (i = 1,2,3), where $\overline{r}_1$, $\overline{r}_2$ and $\overline{r}_3$ are a mutually orthogonal set of unit vectors. (orthonormal) The reference frame $\overline{f}_i$ is similarly defined.

The unit vector $\overline{r}_1$ may be expressed in terms of its three components in the $\overline{f}_i$ reference frame (linearly dependence property) as

$$\overline{r}_1 = a_{11}\overline{f}_1 + a_{12}\overline{f}_2 + a_{13}\overline{f}_3 \tag{B-1}$$

where a_{1j} are real numbers. Taking the scalar dot product of Eq B-1) by $\overline{f}_1$, $\overline{f}_2$, and $\overline{f}_3$ respectively and utilizing Eq (A-5)

$$\overline{f}_i \cdot \overline{f}_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

one obtains

$$\overline{r}_1 \cdot \overline{f}_1 = a_{11}\,\overline{f}_1 \cdot \overline{f}_1 + a_{12}\,\overline{f}_2 \cdot \overline{f}_1 + a_{13}\overline{f}_3 \cdot \overline{f}_1$$

or

$$\begin{aligned} \overline{r}_1 \cdot \overline{f}_1 &= (1)(1)\cos(\overline{r}_1,\overline{f}_1) = a_{11} = \cos\alpha_1 \\ \overline{r}_1 \cdot \overline{f}_2 &= (1)(1)\cos(\overline{r}_1,\overline{f}_2) = a_{12} = \cos\beta_1 \\ \overline{r}_1 \cdot \overline{f}_3 &= (1)(1)\cos(\overline{r}_1,\overline{f}_3) = a_{13} = \cos\gamma_1 \end{aligned} \tag{B-2}$$

where α_1, β_1, and γ_1, are shown in Fig. (1-1)

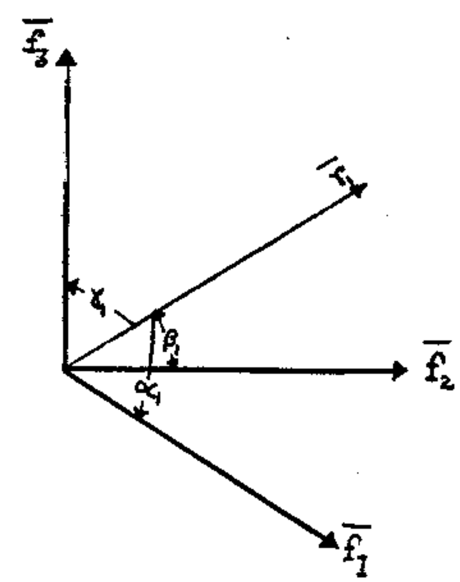


Fig. (B-1)

In brief

$$\overline{r}_1 \cdot \overline{f}_j = a_{1j} \qquad (j = 1,2,3). \tag{B-3}$$

where a_{ij} are called direction cosines of $\overline{r}_1$, since they are equal to the cosines of the angles between the unit vector $\overline{r}_1$ and $\overline{f}_1$, $\overline{f}_2$, and $\overline{f}_3$.

Similar analysis for $\overline{r}_2$ and $\overline{r}_3$ yields

$$\begin{aligned} \overline{r}_2 &= a_{21}\overline{f}_1 + a_{22}\overline{f}_2 + a_{23}\overline{f}_3 \\ \overline{r}_3 &= a_{31}\overline{f}_1 + a_{32}\overline{f}_2 + a_{33}\overline{f}_3 \end{aligned} \tag{B-4}$$

and

$$\begin{aligned} \overline{r}_2 \cdot \overline{f}_j &= a_{2j} \\ \overline{r}_3 \cdot \overline{f}_j &= a_{3j} \end{aligned} \tag{B-5}$$

The nine scalar equations of Eq(B-3) and (B-5) can be written as

$$\overline{r}_i \cdot \overline{f}_j = a_{ij} \quad \begin{matrix} (i = 1, 2,3) \\ (j = 1, 2,3) \end{matrix} \tag{B-6}$$

It is to be noted that the one vector equation, Eq(1-1) yields three scalar equations Eq (B-2).

The three vector equations of Eq(1-1) and (1-4) are

$$\begin{aligned} \overline{r}_1 &= a_{11}\overline{f}_1 + a_{12}\overline{f}_2 + a_{13}\overline{f}_3 = a_{1j}\overline{f}_j \\ \overline{r}_2 &= a_{21}\overline{f}_1 + a_{22}\overline{f}_2 + a_{23}\overline{f}_3 = a_{2j}\overline{f}_j \\ \overline{r}_3 &= a_{31}\overline{f}_1 + a_{32}\overline{f}_2 + a_{33}\overline{f}_3 = a_{3j}\overline{f}_j, \end{aligned} \tag{B-7}$$

or in summation form

$$\overline{r}_i = \sum_{j=1}^{3} a_{ij}\overline{f}_j \qquad (i = 1,2,3). \tag{B-8}$$

USEFUL PROPERTIES OF THE DIRECTION COSINES

Taking the scalar product of Eq(7) by $\overline{r}_1$, $\overline{r}_2$, and $\overline{r}_3$ respectively

$$\begin{aligned} \overline{r}_1 \cdot \overline{r}_1 &= 1 = a_{11}^2 + a_{12}^2 + a_{13}^2 = a_{1j}^2 = \cos(\overline{r}_1, \overline{r}_1) \\ \overline{r}_2 \cdot \overline{r}_2 &= 1 = a_{21}^2 + a_{22}^2 + a_{23}^2 = a_{2j}^2 = \cos(\overline{r}_2, \overline{r}_2) \\ \overline{r}_3 \cdot \overline{r}_3 &= 1 = a_{31}^2 + a_{32}^2 + a_{33}^2 = a_{3j}^2 = \cos(\overline{r}_3, \overline{r}_3) \end{aligned} \tag{B-9}$$

and in brief

$$1 = \sum_{j=1}^{3} a_{ij} \quad (i = 1,2,3) \tag{B-10}$$

which is the well known statement that the sum of the squares of the direction cosines of a line is equal to unity, or, that if the cosine of the angle between two lines is equal to unity, the angle is zero.

Now consider the results of dotting the first equality of Eq(B-7) by $\overline{r}_2$.

$$\overline{r}_1 \cdot \overline{r}_2 = 0 = (a_{11}\overline{f}_1 + a_{12}\overline{f}_2 + a_{13}\overline{f}_3) \cdot (a_{21}\overline{f}_1 + a_{22}\overline{f}_2 + a_{23}\overline{f}_3)$$

$$0 = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = a_{1j}\ a_{2j}, \tag{B-11}$$

similarly

$$\begin{aligned}
\overline{r}_1 \cdot \overline{r}_3 &= 0 = \sum_{j=1}^{3} a_{1j}a_{3j} \\
\overline{r}_2 \cdot \overline{r}_1 &= 0 = \sum_{j=1}^{3} a_{2j}\,a_{1j} \\
\overline{r}_2 \cdot \overline{r}_3 &= 0 = \sum_{j=1}^{3} a_{2j}\,a_{3j} \\
\overline{r}_3 \cdot \overline{r}_1 &= 0 = \sum_{j=1}^{3} a_{3j}\,a_{1j} \\
\overline{r}_3 \cdot \overline{r}_2 &= 0 = \sum_{j=1}^{3} a_{3j}\,a_{2j}
\end{aligned} \tag{B-12}$$

The six scalar equations of Eq(B-11) and (B-12) are the familiar statements that the sum of the products of the direction cosines is equal to zero when two lines are orthogonal.

Consider now some results of taking the vector cross product.

$$\bar{r}_2 \times \bar{r}_3 = \bar{r}_1 = (a_{21}\bar{f}_1 + a_{22}\bar{f}_2 + a_{23}\bar{f}_3) \times (a_{31}\bar{f}_1 + a_{32}\bar{f}_2 + a_{33}\bar{f}_3)$$

by E_1(A-15) $\bar{r}_1 = \begin{vmatrix} \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

expanding

$$\bar{r}_1 = \bar{f}_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \bar{f}_2 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \bar{f}_3 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

and dotting with $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ respectively

$$\bar{r}_1 \cdot \bar{f}_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{11} = a_{22}a \quad -a_{32}a_{23}$$

$$\bar{r}_1 \cdot \bar{f}_2 = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{12} = a_{23}a_{31} - a_{21}a_{33}$$

$$\bar{r}_1 \cdot \bar{f}_3 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{13} = a_{21}a_{32} - a_{31}a_{22}$$

Thus each element of the direction cosine matrix is equal to its cofactor, as can be shown for the other six elements through similar analysis, yielding:

$$\begin{aligned}
a_{11} &= a_{22}a_{33} - a_{23}a_{32} \\
a_{12} &= a_{23}a_{31} - a_{21}a_{33} \\
a_{13} &= a_{21}a_{32} - a_{22}a_{31} \\
a_{21} &= a_{13}a_{32} - a_{12}a_{33} \\
a_{22} &= a_{11}a_{33} - a_{13}a_{31} \\
a_{23} &= a_{12}a_{31} - a_{11}a_{32} \\
a_{31} &= a_{12}a_{23} - a_{13}a_{22} \\
a_{32} &= a_{13}a_{21} - a_{11}a_{23} \\
a_{33} &= a_{11}a_{22} - a_{12}a_{21}
\end{aligned} \qquad \text{(B-13)}$$

The results of equations (B-9),(B-12) and (B-13) will be used in the following two sections.

ELEMENTARY MATRIX THEORY

Let the a_{ij} of Eq. (1) be arranged in the following array:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Such an array which consists of three rows and three columns of numbers (elements) will called a square matrix of order 3.

An array having m rows and n columns will be said to be an $m \times n$ matrix. Two $m \times n$ matrices are defined to be equal if the corresponding elements of the two matrices are equal. That is, if

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}$$

and

$$B = (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \cdots b \end{pmatrix}$$

and if

$$A = B$$

then

$a_{ij} = b_{ij}$ for each i and j, and conversely.

The sum of two $m \times n$ matrices is defined to be an $m \times n$ matrix, each element of which is the sum of the corresponding elements of the two matrices. That is, if

$$A + B = D$$

then

$d_{ij} = a_{ij} + b_{ij}$ for each i and j.

The product, AB, of two matrices is such

if

$$D = AB$$

then
$$d_{ij} = \sum_{r=1}^{n} a_{jr} \; b_{rj},$$

where n is the number of columns of the multiplier.

The following two theorems are given without proof:

(a) Addition of matrices is commutative and associative; that is

$$A + B = B + A$$

$$A + (B + D) = (A + B) + D.$$

(b) Multiplication of matrices is associative but not in general commutative; that is

$$A(BD) = (AB)D$$

$$AB \neq BA \quad \text{(in general)}.$$

Let the square matrix of order n, each element, δ_{ij}, of which is defined by

$$\delta_{ij} = \begin{pmatrix} 1 \text{ if } i = j \\ 0 \text{ if } i \quad j \end{pmatrix}$$

be denoted with the symbol I. This matrix will be called the identity matrix.

The minor, M_{ij}, of the element a_{ij} in an $n \times n$ matrix $A = (a_{ij})$ is the determinant of the $(n - 1) \times (n - 1)$ matrix which remains when elements of the ith row and jth column are deleted from the matrix A. The cofactor of the element a_{ij} is defined to be $(-1)^{i+j} M_{ij}$. Two additional theorems are given without proof:

(c) To each square matrix, A, of order n, there exists a unique matrix, C, such that

$$CA = AC = I$$

providing the determinant of the matrix is not zero. $C = (c_{ij})$ is called the inverse of A.

$$\text{(d)} \quad (c_{ij}) = \left(\frac{A_{ji}}{d(A)} \right) \text{ where}$$

A_{ji} is the cofactor of the element a_{ji} and $d(A)$ is the determinant of the matrix A.

DIRECTION COSINE MATRICES

Using definitions (a) and (c) for equality and multiplications of matrices of the preceding section, Eq. (1) can be written as a matrix equation. For both

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix}$$

and 1 x 3 matrices while the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is a square matrix, and hence

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = A(\bar{f}_i) \tag{B-14}$$

also

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \tag{B-15}$$

where α_i, β_i, γ_i, are direction angles between the $\bar{f}_1$ vector and $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$, the $\bar{f}_2$ vector and $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$, and the $\bar{f}_3$ vector and $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ respectively.

The cofactors of the elements a_{ij} of A are found to be

$$\begin{aligned}
A_{11} &= a_{22}a_{33} - a_{23}a_{32} \\
A_{12} &= a_{23}a_{31} - a_{21}a_{33} \\
A_{13} &= a_{21}a_{32} - a_{31}a_{22} \\
A_{21} &= a_{13}a_{32} - a_{12}a_{32} \\
A_{22} &= a_{11}a_{33} - a_{13}a_{31} \\
A_{23} &= a_{12}a_{31} - a_{11}a_{32} \\
A_{31} &= a_{12}a_{23} - a_{13}a_{22} \\
A_{32} &= a_{13}a_{21} - a_{11}a_{23} \\
A_{33} &= a_{11}a_{22} - a_{12}a_{21}
\end{aligned} \tag{B-16}$$

but by (1-13) $A_{ij} = a_{ij}$.

The determinant d (A) -

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}{}^2 a_{12}{}^2 + a_{13}{}^2 = 1$$

and hence using theorem (d)

$$A^{-1} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

In order to express the vectors $\bar{f}_i$ in terms of $\bar{r}_i$ it is only necessary to multiply both members of Eq's. (B-8) on the left by A^{-1}. That is

$$A^{-1}(\bar{r}_i) = A^{-1}(A\bar{f}_i) = I(\bar{f}_i) = (\bar{f}_i)$$

since $A^{-1}A = I$. Thus

$$\begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \qquad \text{(B-17)}$$

Equations (B-8) and (B-10) are often expressed in the following manner:

	$\bar{f}_1$	$\bar{f}_2$	$\bar{f}_3$
$\bar{r}_1$	a_{11}	a_{12}	a_{13}
$\bar{r}_2$	a_{21}	a_{22}	a_{23}
$\bar{r}_3$	a_{31}	a_{32}	a_{33}

(B-18)

The product of the two matrices to the right in equation (B-17) is obtained by the row column rule as a single column matrix.

$$\begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} = \begin{pmatrix} a_{11}\bar{f}_1 + a_{12}\bar{f}_2 + a_{13}\bar{f}_3 \\ a_{21}\bar{f}_1 + a_{22}\bar{f}_2 + a_{23}\bar{f}_3 \\ a_{31}\bar{f}_1 + a_{32}\bar{f}_2 + a_{33}\bar{f}_3 \end{pmatrix} \qquad \text{(B-19)}$$

Since two matrices are equal if their elements are equal, the three equations of Eq.(B-7) can be obtained from Eq.(B-19) by equating the elements of the two matrices of Eq. (B-19).

It is often necessary to obtain the transformation equations relating the vectors of a third reference to those of a first reference frame when the former frame has been oriented with respect to a second frame and when the second frame was oriented with respect to the first frame. That is, if a transformation, T_1, orients the $\bar{r}$-frame with respect to the $\bar{f}$-frame and a transformation, T_2, orients an $\bar{S}$-frame with respect to the $\bar{r}$-frame, the equations between the vectors $\bar{f}_i$ and $\bar{s}_i$ are easily obtained by matrix multiplication. Let the matrix equation relating the $\bar{s}_i$ to the $\bar{r}_i$ be

$$\begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{12} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{pmatrix} \tag{B-19}$$

If the right member of Eq. (B-14) is substituted for the matrix $(\bar{r}_i)$ in this equation, it follows [redacted]

$$\begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \tag{B-20}$$

or

$$\begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} \tag{B-21}$$

where

$$\begin{aligned}
g_{11} &= \beta_{11} a_{11} + \beta_{12} a_{21} + \beta_{13} a_{31} \\
g_{12} &= \beta_{11} a_{12} + \beta_{12} a_{22} + \beta_{13} a_{32} \\
g_{13} &= \beta_{11} a_{13} + \beta_{12} a_{23} + \beta_{13} a_{33} \\
g_{21} &= \beta_{21} a_{11} + \beta_{22} a_{21} + \beta_{23} a_{31} \\
g_{22} &= \beta_{21} a_{12} + \beta_{22} a_{22} + \beta_{23} a_{32} \\
g_{23} &= \beta_{21} a_{13} + \beta_{22} a_{23} + \beta_{23} a_{33} \\
g_{31} &= \beta_{31} a_{11} + \beta_{32} a_{21} + \beta_{33} a_{31} \\
g_{32} &= \beta_{31} a_{12} + \beta_{32} a_{22} + \beta_{33} a_{32} \\
g_{33} &= \beta_{31} a_{13} + \beta_{32} a_{23} + \beta_{33} a_{33}
\end{aligned}$$

a simple demonstration of why rows and columns can be interchanged for the direction cosine matrices for the orthonormal reference frames used here is as follows

$$\bar{r}_i = (a_{ij})\bar{f}_j \tag{B-22}$$

since the inverse matrix $(a_{ij})^{-1} = (c_{ij})$

then multiplying Eq. (B-22) by (c_{ij})

$$(a_{ij})^{-1}\overline{r}_i = (c_{ij})\overline{r}_i = (a_{ij})(a_{ij}) = I\overline{f}_j = \overline{f}_j \qquad (B\text{-}23)$$

$$\begin{pmatrix} \overline{f}_1 \\ \overline{f}_2 \\ \overline{f}_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \overline{r}_3 \end{pmatrix} = \begin{pmatrix} c_{11}\overline{r}_1 + c_{12}\overline{r}_2 + c_{13}\overline{r}_3 \\ c_{21}\overline{r}_1 + c_{22}\overline{r}_2 + c_{23}\overline{r}_3 \\ c_{31}\overline{r}_1 + c_{32}\overline{r}_2 + c_{33}\overline{r}_3 \end{pmatrix} \qquad (B\text{-}24$$

Equating elements of (B-24), the 1st element

$$\overline{f}_1 = c_{11}\overline{r}_1 + c_{12}\overline{r}_2 + c_{13}\overline{r}_3 \qquad (B\text{-}25)$$

Dotting Eq. (B-25) by $\overline{r}_1$

$$\overline{f}_1 \cdot \overline{r}_1 = a_{11} = c_{11}.$$

Equating the second elements

$$\overline{f}_2 = c_{21}\overline{r}_1 + c_{22}\overline{r}_2 + c_{23}\overline{r}_3$$

and dotting by $\overline{r}_1$

$$\overline{f}_2 \cdot \overline{r}_1 = c_{21} = a_{12}$$

by Eq. (B-4).

and similarly for the other elements

$$c_{ij} = a_{ji}.$$

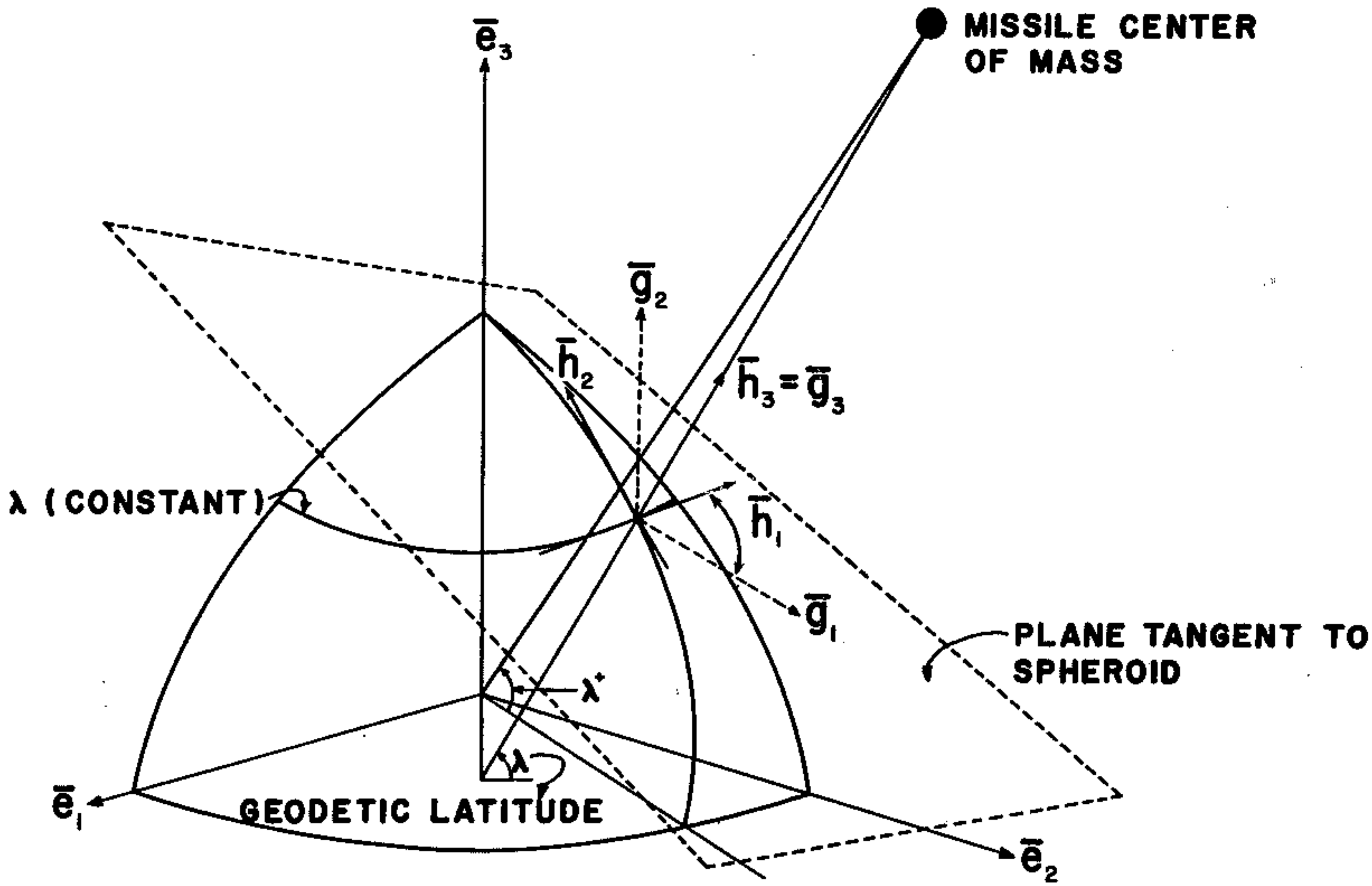
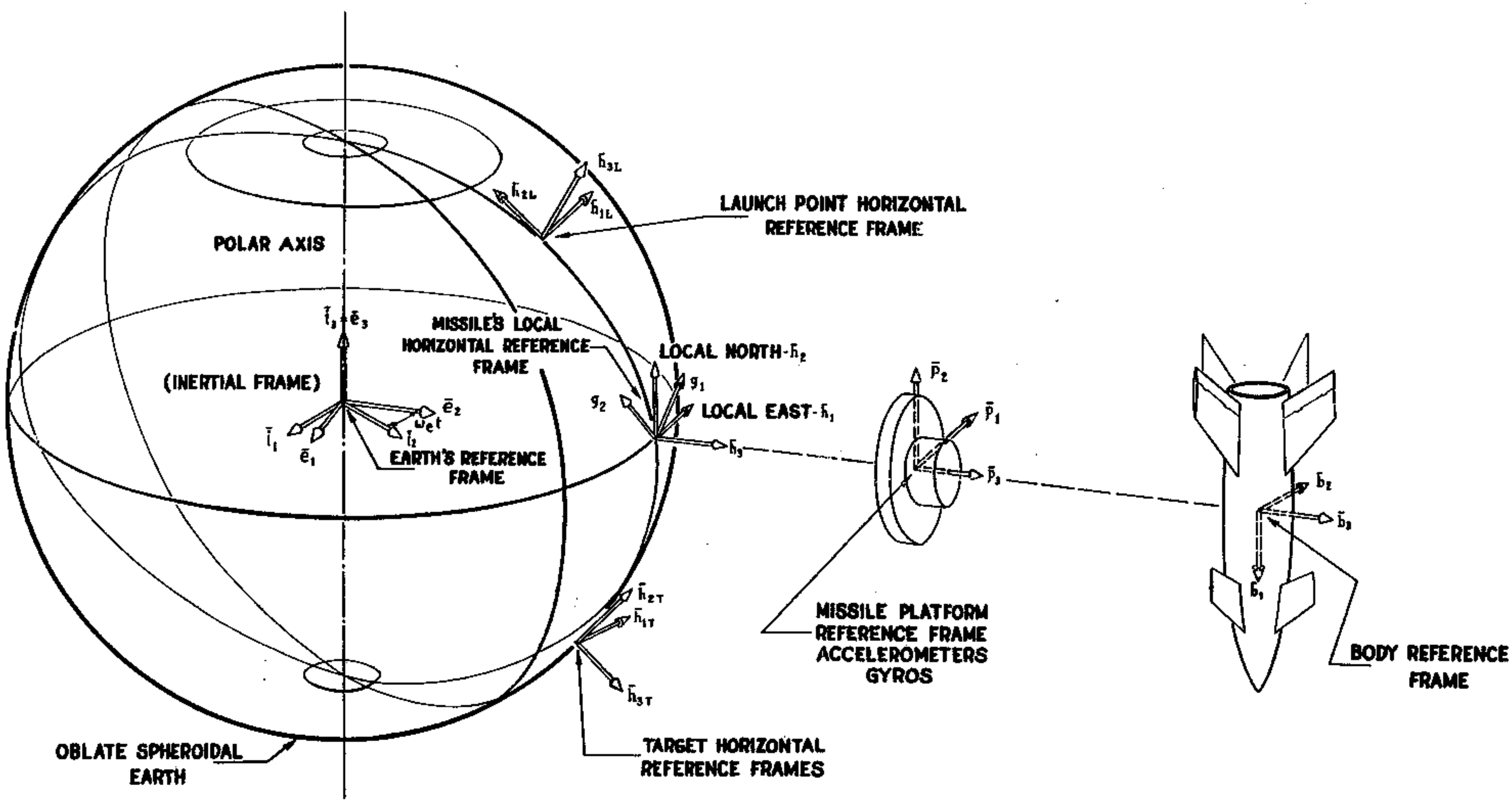


Fig. 15. Local Vertical Reference Frames.



REFERENCE FRAMES

FIG. 16.